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Electrical Engineering
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UDGIVET VED
UDVALGET FOR INGENIØRVIDENSKABELIG FORSKNING

THE PROPAGATION OF RADIO WAVES

ALONG THE SURFACE OF THE EARTH AND IN THE ATMOSPHERE

BY

P. O. PEDERSEN

COPENHAGEN

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1927

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Dedicated to my Friend

DR. VALDEMAR POULSEN

who twenty years ago and first of all demonstrated perfect radio telephony.

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PREFACE.

In this book I have attempted to give a connected physical theory of radio wave propagation. The main basis of such a theory must be the state of ionization and the electrical and optical properties of the atmosphere and it has therefore been necessary to discuss, at considerable length, such questions as air pressure at high altitudes, mean values of free path for the electrons and ions, sources of radiation etc.

It has also been necessary to determine the influence of free electrons and of ions on the conductivity and on the dielectric constant of the air and to consider the influence of the magnetic field of the earth on these quantities and on the rotation of the plane of polarization.

In working out the necessary formulas for the influence of electrons and ions I have not aimed at mathematical elegance and brevity but have tried to carry out the calculations in such a way that the physical aspect of the question is never lost sight of.

However, it is not the conductivity and the dielectric constant but rather the refractive index and the attenuation constant with which we are directly concerned in radio wave propagation, and the refractive index cannot in general be set equal to the square root of the dielectric constant, but must be determined from the conductivity and the dielectric constant by means of the well known but rather cumbrous formulas, — a fact overlooked by some authors.

Radio engineers and others who are more interested in the results themselves than in the derivation of results may omit this material and forthwith start at chapter XI, going back to the earlier chapters only now and then as necessity arises.

The development in this field has lately been so rapid that a great part of the most important papers bearing on the subject have appeared during the time taken in the printing of this book. I have endeavoured to take account of as much of this recent work as possible, even at the risk of disturbing, at least in some places, the logical order of the treatment.

I may, however, have overlooked important publications which ought to have been mentioned. The number of papers on this subject is so great that it is impossible to make a thorough study of them all.

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I wish to thank the »*H. C. Ørsted*-Foundation« — founded by the Great Northern Telegraph Company — for its liberal grants to defray expenses for assistance in carrying out the great bulk of numerical calculations which have been necessary, and in preparing this work for the press. I also wish to thank »*Danmarks Naturvidenskabelige Samfund*« for taking over the publication of this book.

Especial thanks are due to my assistants, Mr. *J. P. Christensen* and Mr. *J. Rybner* for their invaluable help and criticism. Mr. *Rybner* designed the various nomograms and also had charge of all the calculations, in which task he was ably assisted by Mr. *T. Bjerge* and Mr. *T. Krarup*.

My friend and former pupil Mr. *Axel G. Jensen* has shown me the great kindness to look through the whole MS and correct some of the linguistic shortcomings.

Copenhagen, June 1927.

P. O. PEDERSEN.

Most of the results contained in the following have been presented in a series of lectures before various societies of Copenhagen, namely:

Before »*Selskabet for Naturlærens Udbredelse*« on October 20-th, 1926.
» »*Danmarks naturvidenskabelige Samfund*« on November 22-nd, 1926.
» »*Det kgl. danske Videnskabernes Selskab*« on December 3-rd, 1926.
» »*Elektroteknisk Forening*« on December 10-th, 1926.
» »*Fysisk Forening*« on February 14-th, 21-th and 28-th, 1927.

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CHAPTER I.

THE PROPAGATION OF RADIO WAVES ALONG THE SURFACE OF THE EARTH AND IN THE ATMOSPHERE.

INTRODUCTION.

Ever since the very first days of radio telegraphy, the question of the manner of propagation of electro-magnetic waves along the surface of the earth has been occupying the minds of a large number of mathematicians, physicists and radio engineers. As a result thereof a considerable literature has developed concerning this very interesting and important problem. The interest in the problem has increased still more during the last few years, *i. e.* since the long-range ability of the short waves has become generally acknowledged, and it is now almost impossible to read over even the more important of the papers published on the subject.

The literature may as a whole be divided into two groups, namely one group consisting of the investigations of actual conditions — that is the determination of the intensity of the arriving waves and possibly of their direction and state of polarization — and another group consisting of the theoretical papers having for their object the presentation of a reasonable explanation of the results found in practice.

Neither one of these two problems can yet be regarded as solved in a satisfactory manner. The farthest steps appear to have been made towards the solution of the first mentioned problem, and the results gained in practice are in general fairly reliable and certain, at any rate for day-transmission on the long waves. For short waves, however, there is still quite a considerable uncertainty concerning their actual propagation and other properties. It should be added, however, that the experimental material is increasing very rapidly, especially concerning the short waves. Many contributions in this direction have been made by amateurs.

The entire problem of transmission, however, is so complicated that it is very difficult to attempt an explanation of the conditions solely by means of experiments and measurements. A tolerably well founded theory, suitable as a guide, might here prove very useful. But such a theory can hardly be said to exist at present, although essential contributions towards the same have been presented especially during the last few months and while this work has been in preparation for the press. It would indeed be very difficult, not to say impossible, to build up a theory capable of explaining at the same time all the highly varying facts found in practice. The present work is an attempt to advance the theory of propagation of waves another small step, partly by demonstrating the invalidity of certain of the earlier made assumptions, and

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partly by giving a more detailed and collective treatment of some of the more important conditions governing the propagation of waves.

Although the picture of the propagation of radio waves as given below would hardly be the final one in every detail, we still hope that the present work will prove an aid towards the further development of the theory of their propagation.

Before going over to the more detailed treatment of all the conditions affecting the problem of transmission, we shall first very briefly consider the propagation of radio waves under simplifying and highly idealized conditions. We assume that a vertical transmitting antenna of effective height H (see Fig. 1) radiates waves of the length λ, and carries the effective current I_s at the base. We make the further assumption that the antenna is erected on a perfectly conducting infinite plane surface *mn*. In that case, as demonstrated by *H. Hertz*[1], the electrical field at a great distance from the antenna will be perpendicular to the surface *mn*, and its effective value E_r, at the distance *r*, will be determined by

Fig. I. 1. A Vertical Antenna on a perfectly Conducting Plane Surface.

$$E_r = 1.2 \cdot 10^{-3} \pi \left(\frac{H}{\lambda}\right) \cdot \frac{1}{r} \cdot I_s \quad (\text{volt} \cdot \text{cm}^{-1};\ \text{r in km}) \tag{1}$$

Consequently:

$$\frac{E_r}{E_{100}} = \frac{100}{r} \tag{2}$$

The values of this ratio are plotted in Fig. 3 for the range $r = 100$ km to $r = 20\,000$ km.

At a large distance from the antenna the electromagnetic energy moves in the direction of radius vector and the intensity of radiation is proportional to $\sin^2 \varphi$, φ being the angle between the axis of the antenna and the direction of the ray (see Fig. 2 where the length of the arrows indicates the intensity of

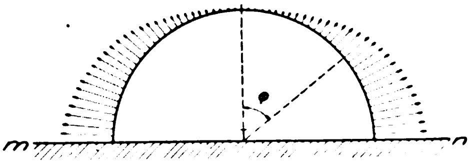

Fig. I. 2. A Diagram showing the Intensity of Radiation at a great Distance from the Antenna in Fig. 1.

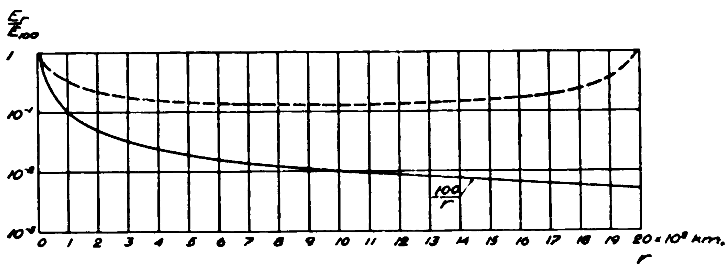

Fig. I. 3. The full-line curve indicates the Attenuation of the Waves propagated along a perfectly conducting plane Surface. The dotted curve indicates the Attenuation, when the Waves travel within the Spherical Shell shown in Fig. 4.

[1] *H. Hertz*: Wied Ann. Bd. 36. p. 1—22, 1889.

radiation). It appears from the figure that the dispersion of the energy of radiation into space increases with the distance from the antenna.

But the earth is not flat, nor is it perfectly conducting, and it is therefore not to be expected that the field intensity along the surface of the earth would vary in the manner indicated by (2).

Even if the earth were a perfect conductor, its shape would still cause a large portion of the energy emitted from the antenna to be lost by radiation into free space, while only a smaller portion of the energy would follow the surface of the earth. The field must consequently decrease more rapidly along the surface of the sphere than is required by formula (2). This feature is further treated in Chapter II, and the reason for calling attention to it here is that now and then, even very recently, the opinion has been set forth that the radio waves simply by themselves follow the surface of the earth during their propagation, in the same manner as alternating electrical currents move along conductors. See, for instance, *J. E. Taylor*'s contribution to the discussion of *J. Hollingworth*'s paper: »The Propagation of Radio Waves, Journ. Inst. El. Eng. Vol. 64, p. 79—589, May 1926«. p. 591 and *Hollingworth*'s reply thereto (l. c., p. 594). *Taylor* appears not to pay due regard to the very large difference, in the two cases, in the concentration of field energy in the vicinity of the surface of the conductor. In propagation along a plane conducting surface, assuming equation (1) to be correct, there will be present, within a distance of 1 cm from the surface of the conductor (the earth), only 1.5×10^{-7} of the entire field energy at the moderate distance of only 100 km from the transmitter — a distance within which the correctness of equation (1), for not too short waves and for good conducting soil, has been established by experiments[1]. But on the other hand, for a 2 mm metallic return circuit the wires of which are spaced 200 mm apart, 0.45 of the entire field energy will be present within a distance of 1 cm from the wires. Since the electric force in the immediate neighbourhood of the surface of a good conductor must necessarily be perpendicular to the latter, or tilting slightly forward in the direction of the flow of energy, it follows that the energy flow must also mainly follow the wires, even if the latter be slightly curved. (*Hollingworth* appears to be of the opinion that the wires lose their ability to guide and collect the field energy, when their conductivity increases without limit, l. c., p. 594).

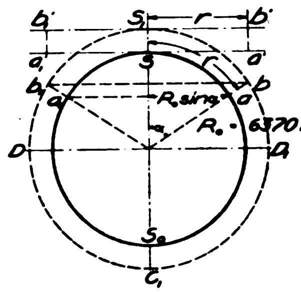

Fig. 4. The Earth surrounded by a Conducting Spherical Shell $S_1DC_1D_1$.

Assuming now on the other hand that the earth is enclosed in a perfectly conducting shell, as $S_1DC_1D_1$ in Fig. 4, then the entire energy radiated from the transmitter will be kept enclosed between the surface of the earth and the shell. The energy passing the ring aba_1b_1 between the two spherical surfaces must then be very nearly equal to the energy which would pass the cylinder $a'b'a_1'b_1'$, if the radiation took place between the two plane conducting surfaces $a_1'Sa'$ and $b_1'S_1b'$. In that case the equation (2) would be replaced by

[1] See for instance: *M. Abraham:* Wied. Ann. Bd. 66, p. 435—472, 1898. Ann. d. Physik. Bd. 2, p. 32—61, 1900. Phys. Zeitschr. Bd. 2, p. 329—334, 1901.
W. Duddell and *J. E. Taylor:* Journ. Inst. El. Ing. Vol. 35, p. 321, 1905.
M. Reich: Phys. Zeitschr. Bd. 14, p. 934—938, 1913.

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$$\frac{\bar{E}_r}{E_{100}} = \frac{10}{\sqrt{r}}\sqrt{\frac{\alpha}{\sin\alpha}} = \frac{10}{\sqrt{6370\,\alpha}}\sqrt{\frac{\alpha}{\sin\alpha}} = \frac{0.125}{\sqrt{\sin\alpha}} \tag{3}$$

where $\bar{E}_r$ is the effective mean value of the electric field in the ring aba_1b_1 at the distance r from the transmitter.

$\bar{E}_r$ must obviously decrease less rapidly than E_r as calculated from the formula (2). While E_r decreases continuously with increasing r, $\bar{E}_r$ will reach a minimum value for $r = 10\,000\,km$, as indicated by the dotted line in Fig. 3, and will then increase again towards the antipole S_0 (see Fig. 4), in the same manner as it decreased from the transmitter S, the intensity being the same at points located symmetrically with respect to the diametrical plane DD_1.

Although the formulas (1) and (2), for short distances and for long waves, agree quite well with experience and although the conditions in the vicinity of the antipole agree with equation (3), in so far as it has been possible in several cases to observe an increase of the intensity of signals, when the antipole is approached[1], none of these formulas will give even an approximately correct representation of the actual conditions. Thus, according to these formulas, the relative field-intensities would be independent of the wave length; on the contrary experience shows that these intensities depend very much upon the wave length, being, for instance, at great distances from the transmitter as much as 10^{20} times as large for one wave length as for another. Finally, even when the same wave length is used, the conditions are by no means constant, but on the contrary very highly varying. The received field-intensity is thus widely different between summer and winter, day and night, nay even from minute to minute large variations may occur.

The cause of this must be looked for partly in the fact that the earth is not a perfect conductor and partly and mainly in the fact that the atmosphere is not a perfect insulator with uniform dielectric constant. The very large and very rapid variations in field-intensity observed so frequently at great distances from the transmitter must undoubtedly be due to changes in the state of the atmosphere, and a theory of wave propagation not taking the atmospheric conditions into account must therefore be considered unsatisfactory to start with. This applies for instance to the theoretical considerations mentioned in the following Chapters II and III. These considerations are therefore treated only to such an extent as may be useful for the following investigations which principally deal with the influence of the atmosphere, although also taking into consideration the nature of the ground along which the waves are travelling.

It is certainly impossible, at the present time, to build up a theory of the propagation of radio waves that is satisfactory in all details. Our knowledge of the propagation of waves is as yet too small for this, and especially is our knowledge of atmospheric conditions at great heights above the surface of the ground as yet far too insufficient and uncertain to enable us to do so. Furthermore, the physical basis for a reliable theoretical treatment of the state of the atmosphere at great heights is in many respects very uncertain, as it will be shown further in the following.

A rational extention of the theory of propagation of radio waves will require a further clarification of the physical basis, and preliminary experiments in that direction are being prepared. On the other hand, if the

[1] See for instance *M. Tranier:* L'Onde Électrique, 3e année, p. 70—82, 142—152, 1924.

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physical basis is secure, then it will be possible by suitable investigations of the *propagation of radio waves*, and by means of the considerations set forth in the following, to form a relatively reliable picture of the conditions of pressure and ionization in the *upper atmosphere*, and at the same time we shall then be able to build up a fairly detailed theory of the manner of propagation of the waves. It is our hope that the present work may have advanced this development in some respects and may be of assistance in its further advancement.

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CHAPTER II.

THE PROPAGATION OF ELECTRO-MAGNETIC WAVES ALONG THE SURFACE OF A SPHERE.

1. *A Sphere surrounded by a Homogeneous, Non-Conducting Medium having a Dielectric Constant equal to Unity.*

In attempting an explanation of the propagation of radio waves over the surface of the earth it would seem natural to assume the earth replaced by a homogeneous sphere of radius 6370 km and the surrounding atmosphere replaced by a homogeneous non-conducting medium having a dielectric constant equal to unity, and theoretical physicists started to solve the thus simplified problem immediately after the demonstration by *Marconi* of the possibility of radio telegraphy over long distances. In spite of its physical simplicity, however, this problem still presents many mathematical difficulties, which only recently have been substantially overcome. We shall not here enter into any detailed discussion of this question[1], but merely note the conclusion, namely that the ratio between the electrical field intensities at distances r and r_0 from the transmitter, *i. e.* the attenuation factor for the amplitude at a distance $(r - r_0)$, under the above mentioned assumptions is determined by:

---

[1] An excellent summary of the literature up to 1920 may be found in:

L. Bouthillon: La propagation des ondes életromagnétiques à la surface de la terre. (Librairie Delagrave. Paris 1921).

There may also be mentioned:

H. M. Macdonald: Proc Roy. Soc. (A). Vol. 71, p. 51, 1903; Vol. 72, p. 59, 1904; Phil. Trans. Roy. Soc. (A). Vol. 210, p. 113, 1911; Proc Roy. Soc. (A). Vol. 90, p. 50, 1914; Vol. 92, p. 493, 1916; Vol. 98, p. 216, 1920.

Lord Rayleigh: Proc. Roy. Soc. (A). Vol. 72, p. 44, 1904; Jahrbuch d. drahtl. Telegraphie. Bd. 3, p. 445, 1910; Rend. d. circ. mat. de Palermo, pp. 169—261, 1910.

J. W. Nicholson: Phil. Mag. (6), Vol. 19, pp. 276, 435, 516, 757, 1910; Vol. 20, p. 157, 1910; Vol. 21, pp. 62, 281, 1911.

H. W. March: Ann. d. Physik (4), Bd. 37, p. 29, 1912.

W. v. Rybczinski: Ann. d. Physik (4), Bd. 41, p. 191, 1913.

A. E. H. Love: Phil. Trans. Roy. Soc. (A). Vol. 215, p. 105, 1915.

G. N. Watson: Proc. Roy. Soc. (A). Vol. 95. pp. 83, 546, 1919.

Balth. van der Pol, jun.: Phil. Mag. (6). Vol. 38, p. 365, 1919.

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$$\frac{E_r}{E_{r_0}} = \sqrt{\frac{\sin \alpha_0}{\sin \alpha}} \cdot e^{\frac{-0.00375\,(r - r_0)}{\lambda^{1/3}}}, \quad [\text{r and } \lambda \text{ in km}] \tag{1}$$

where $\alpha = \frac{r}{6370}$ is the central angle corresponding to the distance r.

If two different wave lengths, λ_1 and λ_2, are used for the transmission, the ratio f between the factors of attenuation in the two cases will be, according to (1):

$$f = e^{-0.00375\,(r - r_0)\left(\frac{1}{\lambda_2^{1/3}} - \frac{1}{\lambda_1^{1/3}}\right)} \tag{2}$$

Example: for $r - r_0 = 10\,000$ km, $\lambda_1 = 8$ km and $\lambda_2 = 0.027$ km the formula (2) gives:

$$f = 0.7 \cdot 10^{-46}$$

Experience shows, however, that for at least a considerable portion of every 24 hour period the short wave will be attenuated at any rate not more than the long wave by propagation over this distance. It is therefore evident that the theory referred to does not at all give an approximate representation of the actual conditions. Furthermore, we know that even the longer waves are attenuated to an extent far less than that required by the theory, and finally, this theory does not at all give any explanation of the highly varying values of the attenuation. The assumptions forming the basis of the theory are therefore not confirmed by experience.

In the foregoing, the assumption has been made that the sphere is a perfect conductor, but *Love, Macdonald* and *Watson* in their works mentioned above have demonstrated that the result will be much the same, even if the conductivity of the sphere is taken to be equal to that of ocean water or of dry earth.

The calculations referred to above are quite complicated and not very easy to follow and they are therefore not suited to form part of a more elementary theory based on somewhat modified assumptions.

2. *A perfectly Conducting Sphere Surrounded by a Homogeneous Non-Conductive Layer of Air, limited by an Outer Spherical Shell the Dielectric Constant of which is Different from that of the Inner Layer of Air.*

This case has been treated by *H. M. Macdonald*[1], under the assumption that the dielectric constants of the layer of air and of the spherical shell are invariable. However, for the reasons mentioned in the Introduction and above under heading 1, this basis is also unfit as a starting point for a fertile theory of the propagation of radio waves, however interesting these investigations may be from a theoretical point of view[2].

[1] *H. M. Macdonald:* Proc. Roy. Soc. (A). Vol. 108, p. 52, 1925.

[2] From a letter from *Mary Taylor* published quite recently in »Nature« (p. 791, June 5, 1926) it seems to appear that new results may be expected on the theoretical treatment of the problems mentioned in the present chapter. But no satisfactory solution of the problem of propagation of radio waves can be reached in this way, as long as the assumptions underlying the theory are not modified essentially. First of all it will be necessary to pay proper attention to the influence of the atmosphere.

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CHAPTER III.

THE PROPAGATION OF ELECTRO-MAGNETIC WAVES ALONG PLANE SURFACES.

1. *Maxwell's Equations for the Electro-magnetic Field.*

The following symbols will be used below:

$f = \frac{\omega}{2\pi}$ = the frequency of the waves.

$c = 3 \cdot 10^{10}$ cm sec^{-1} = the velocity of propagation of electro-magnetic waves in free space.

$\lambda = \frac{2\pi c}{\omega}$ = the wave length (in cm) in free space.

ε = the dielectric constant (equal to unity in free space).

μ = the permeability (equal to unity in free space).

σ = the conductivity in electro-magnetic c. g. s. units (indicated in the following by e. m. u). (1 e. m. u. $= 1 \cdot 10^9$ ohm^{-1} cm^{-1}).

E = the electric field intensity of the waves. The components along the axes of the rectangular system of co-ordinates used are indicated by E_x, E_y, E_z. E is measured in e. m. u. where nothing else is expressly stated, (1 e. m. u. $= 1 \cdot 10^{-8}$ volt$\cdot$cm^{-1}).

H = the magnetic field intensity of the waves, H_x, H_y and H_z the components of same. H is measured in Gausses.

Maxwell's equations for the electro-magnetic field may then be written:

$$\left.\begin{aligned} -\frac{\partial(\mu H_x)}{\partial t} &= \frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}, \\ -\frac{\partial(\mu H_y)}{\partial t} &= \frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x}, \\ -\frac{\partial(\mu H_z)}{\partial t} &= \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}, \end{aligned}\right\} \quad (I)$$

and

$$\left.\begin{aligned} 4\pi\sigma E_x + \frac{\varepsilon}{c^2}\frac{\partial E_x}{\partial t} &= \frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z}, \\ 4\pi\sigma E_y + \frac{\varepsilon}{c^2}\frac{\partial E_y}{\partial t} &= \frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x}, \\ 4\pi\sigma E_z + \frac{\varepsilon}{c^2}\frac{\partial E_z}{\partial t} &= \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}. \end{aligned}\right\} \quad (II)$$

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It is well known that in the transition from one medium to another the components of the electric and the magnetic fields parallel to the boundary surface have the same value on either side of this surface. The normal components of the dielectric displacement εE_n and of the magnetic induction μH_n pass unaltered through the surface, provided however, in case of the dielectric displacement, that no free electricity be present at the boundary surface.

In the following we assume the permeability to be equal to unity throughout.

2. *Propagation of Plane Waves along a Plane Conducting Surface.*

For one of the two media (the air) the dielectric constant is ε_0 and the specific conductivity σ_0, while for the other (the earth) the dielectric constant is ε and the conductivity σ.

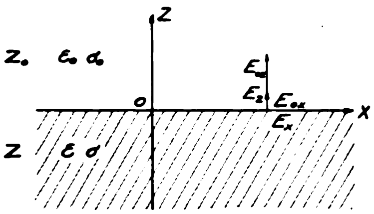

Fig. III. 1. Symbols and System of Co-ordinates: the positive Direction of the Axis of Y pointing towards the Plane of the Paper.

The wave is supposed to move in the positive direction of the X-axis (see Fig. 1) while the Z-axis is perpendicular to the boundary surface. It is shown by *J. Zenneck*[1] that the amplitudes of the waves will then be attenuated at the rate of $e^{-\alpha x}$, $e^{\gamma_0 z}$, and $e^{\gamma z}$ in the direction of X, Z and $-Z$ respectively, where α, γ_0 and γ are determined by the following formulas:

$$\left.\begin{aligned}
s = \alpha + j\beta &= \sqrt{j4\pi\omega\sigma \frac{\left(1 + j\frac{\omega}{\sigma}\frac{\varepsilon}{4\pi c^2}\right)\left(\frac{\sigma_0}{\sigma} + j\frac{\omega}{\sigma}\frac{\varepsilon_0}{4\pi c^2}\right)}{1 + \frac{\sigma_0}{\sigma} + j\frac{\omega}{\sigma}\frac{\varepsilon + \varepsilon_0}{4\pi c^2}}}, \\
r_0 = \gamma_0 + j\delta_0 &= \sqrt{j4\pi\omega\sigma \frac{\left(\frac{\sigma_0}{\sigma} + j\frac{\omega}{\sigma}\frac{\varepsilon_0}{4\pi c^2}\right)^2}{1 + \frac{\sigma_0}{\sigma} + j\frac{\omega}{\sigma}\frac{\varepsilon + \varepsilon_0}{4\pi c^2}}}, \\
r = \gamma + j\delta &= \sqrt{j4\pi\omega\sigma \frac{\left(1 + j\frac{\omega}{\sigma}\frac{\varepsilon}{4\pi c^2}\right)^2}{1 + \frac{\sigma_0}{\sigma} + j\frac{\omega}{\sigma}\frac{\varepsilon + \varepsilon_0}{4\pi c^2}}},
\end{aligned}\right\} \quad (\mathrm{cm}^{-1}) \quad (1)$$

the parameters s, r_0 and r having to satisfy the equations

$$\left.\begin{aligned}
r_0^2 + s^2 &= j\omega\left(4\pi\sigma_0 + j\omega\frac{\varepsilon_0}{c^2}\right) = p_0^2, \\
r^2 + s^2 &= j\omega\left(4\pi\sigma + j\omega\frac{\varepsilon}{c^2}\right) = p^2, \\
r_0 p^2 - r p_0^2 &= 0.
\end{aligned}\right\} \quad (1')$$

For the components of the electric field-intensity at the boundary surface, we have:

[1] *J. Zenneck:* Ann. der Physik. (4). Bd. 23, p. 846, 1907.

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$$\left.\begin{aligned}
&\left(\frac{E_{0x}}{E_{0z}}\right)_{z=0} = \left(\frac{E_x}{E_{0z}}\right)_{z=0} = -\sqrt{\frac{\frac{\sigma_0}{\sigma} + j\,\frac{\omega}{\sigma}\,\frac{\varepsilon_0}{4\pi c^2}}{1 + j\,\frac{\omega}{\sigma}\,\frac{\varepsilon}{4\pi c^2}}}\,,\\
&\left(\frac{E_z}{E_{0z}}\right)_{z=0} = \frac{\frac{\sigma_0}{\sigma} + j\,\frac{\omega}{\sigma}\,\frac{\varepsilon_0}{4\pi c^2}}{1 + j\,\frac{\omega}{\sigma}\,\frac{\varepsilon}{4\pi c^2}}\,.\\
&\left(\frac{|E_x|^2 + |E_z|^2}{|E_{0x}|^2 + |E_{0z}|^2}\right)_{z=0} = \sqrt{\frac{\left(\frac{\sigma_0}{\sigma}\right)^2 + \left(\frac{\omega}{\sigma}\,\frac{\varepsilon_0}{4\pi c^2}\right)^2}{1 + \left(\frac{\omega}{\sigma}\,\frac{\varepsilon}{4\pi c^2}\right)^2}}\,.
\end{aligned}\right\} \qquad (2)$$

The equations (1) and (2) are somewhat simplified by assuming

$$\sigma_0 = 0;$$

and further simplification is obtained by using the two systems of approximate formulas tabulated below and valid, respectively for $\frac{\varepsilon}{4\pi c^2} \ll \frac{\sigma}{\omega}$ (or $\frac{\varepsilon}{2c\sigma\lambda} \ll 1$, [$\lambda$ in cm]), and for $\frac{\varepsilon}{4\pi c^2} \gg \frac{\sigma}{\omega}$ (or $\frac{\varepsilon}{2c\sigma\lambda} \gg 1$, [$\lambda$ in cm]).

Table 1. Approximate Formulas. ($\sigma_0 = 0$; $\varepsilon_0 \leqq \varepsilon$).

	$\frac{\omega}{\sigma}\frac{\varepsilon}{4\pi c^2} \ll 1$	$\frac{\omega}{\sigma}\frac{\varepsilon}{4\pi c^2} \gg 1$	
$\alpha =$	$\frac{\varepsilon_0\sqrt{\varepsilon_0}}{8\pi c^3}\frac{\omega^2}{\sigma}$	$2\pi c\sigma\,\frac{\varepsilon_0\sqrt{\varepsilon_0}}{(\varepsilon+\varepsilon_0)\sqrt{\varepsilon(\varepsilon+\varepsilon_0)}}$	cm^{-1}
$\beta =$	$\frac{\omega}{c}\sqrt{\varepsilon_0}$	$\frac{\omega}{c}\frac{\sqrt{\varepsilon\varepsilon_0}}{\sqrt{\varepsilon+\varepsilon_0}}$	cm^{-1}
$\gamma_0 =$	$-\varepsilon_0\frac{\omega}{c^2}\sqrt{\frac{\omega}{8\pi\sigma}}$	$-2\pi c\sigma\,\frac{\varepsilon_0}{(\varepsilon+\varepsilon_0)\sqrt{\varepsilon+\varepsilon_0}}$	cm^{-1}
$\delta_0 =$	$\varepsilon_0\frac{\omega}{c^2}\sqrt{\frac{\omega}{8\pi\sigma}}$	$\frac{\omega}{c}\frac{\varepsilon_0}{\sqrt{\varepsilon+\varepsilon_0}}$	cm^{-1}
$\gamma =$	$\sqrt{2\pi\sigma\omega}$	$2\pi c\sigma\,\frac{\varepsilon+2\varepsilon_0}{(\varepsilon+\varepsilon_0)\sqrt{\varepsilon+\varepsilon_0}}$	cm^{-1}
$\delta =$	$\sqrt{2\pi\sigma\omega}$	$\frac{\omega}{c}\frac{\varepsilon}{\sqrt{\varepsilon+\varepsilon_0}}$	cm^{-1}
$\left\lvert\frac{E_{0x}}{E_{0z}}\right\rvert_{z=0} = \left\lvert\frac{E_x}{E_{0z}}\right\rvert_{z=0} =$	$\sqrt{\frac{\omega}{\sigma}\frac{\varepsilon_0}{4\pi c^2}}$	$\sqrt{\frac{\varepsilon_0}{\varepsilon}}$	
$\left\lvert\frac{E_z}{E_{0z}}\right\rvert_{z=0} =$	$\frac{\omega}{\sigma}\frac{\varepsilon_0}{4\pi c^2}$	$\frac{\varepsilon_0}{\varepsilon}$	
$\left(\frac{\lvert E_x\rvert^2+\lvert E_z\rvert^2}{\lvert E_{0x}\rvert^2+\lvert E_{0z}\rvert^2}\right)_{z=0} =$	$\frac{\omega}{\sigma}\frac{\varepsilon_0}{4\pi c^2}$	$\frac{\varepsilon_0}{\varepsilon}$	

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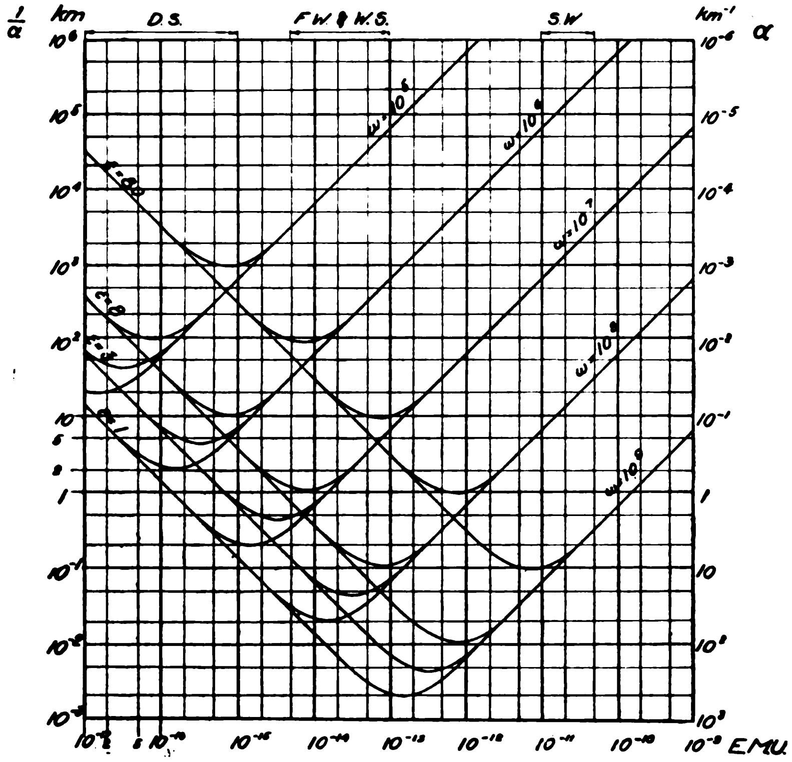

Fig. III. 2. Horizontal Range of the Waves, according to *Zenneck*'s Formulas.

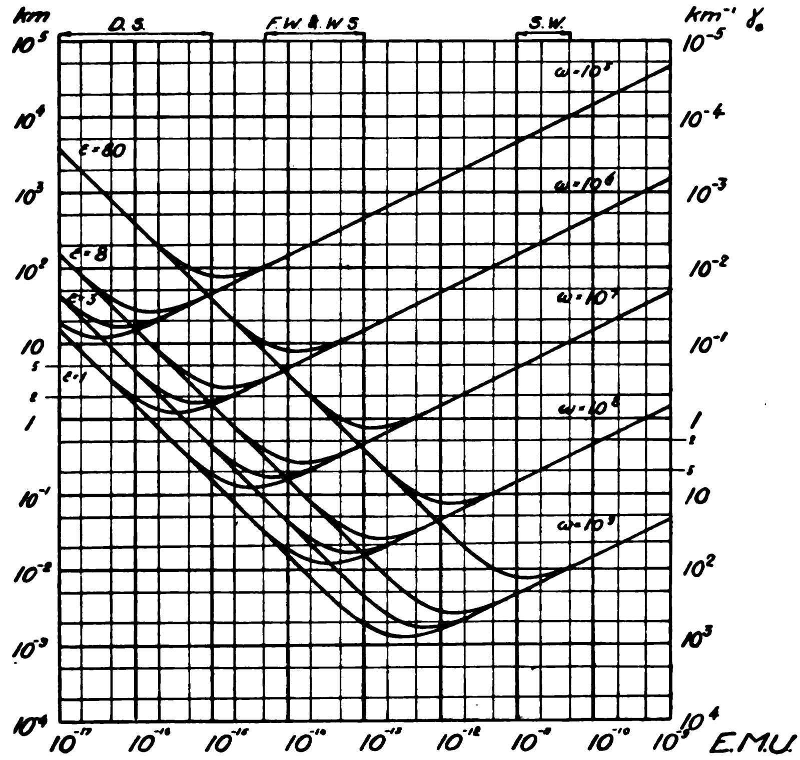

Fig. III. 3. Upward Range of the Waves, according to *Zenneck*'s Formulas.

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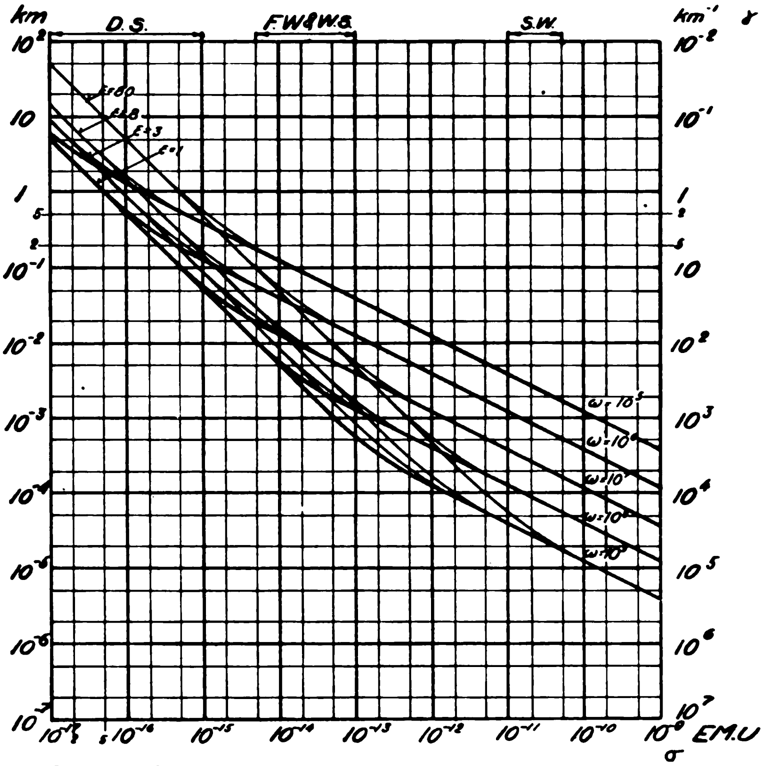

Fig. III. 4. Downward Range of the Waves, according to *Zenneck*'s Formulas.

The values in km of $\frac{1}{\alpha}$, $\frac{1}{\gamma_0}$, and $\frac{1}{\gamma}$ as functions of σ are shown in respectively Figs. 2, 3 and 4[1], for different values of ω and of ε ($\varepsilon_0 = 1$). The value af $\frac{1}{\alpha}$ indicates the distance travelled by the wave, when its amplitude has decreased in the ratio of $1 : e = 0.368$. Correspondingly $\frac{1}{\gamma_0}$ and $\frac{1}{\gamma}$ indicate the distances from the boundary surface to the two planes parallel to the surface and respectively above and below it, in which the amplitude of the waves is $1 : e$ of the amplitude at the boundary surface.

In the following, $\frac{1}{\alpha}$, $\frac{1}{\gamma_0}$ and $\frac{1}{\gamma}$ will occasionally be called the range of the waves in the directions along the boundary surface and normal to this surface into the two media, respectively.

When comparing the values from Fig. 2 with the results of experience we find that the formulas (1) give too small attenuation values for very long waves, especially for large values of σ, while on the other band the same formulas give altogether too great values of the attenuation for the short waves. For transmission over fairly good conducting ground the formula requires

[1]) In *Zenneck*'s Fig. 8, corresponding to our Fig. 4, the values of $\frac{1}{\gamma}$ for very small values of σ converge towards a constant limit, which is not correct.

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that the attenuation constant α be proportional to the second power of the frequency, but this is not true for long waves, since experience has shown that for such waves α will normally be approximately proportional to the square root of the frequency, and it does not at all apply to short waves for which α decreases with increasing frequency, at any rate within rather wide limits.

When, in spite of this pronounced discrepancy, we enter somewhat further into the *Zenneck* theory, the reason is that with certain modifications this theory may be used in an elementary treatment of the propagation of long waves. For use in the following considerations we are therefore going to examine the state of energy for the wave motion concerned.

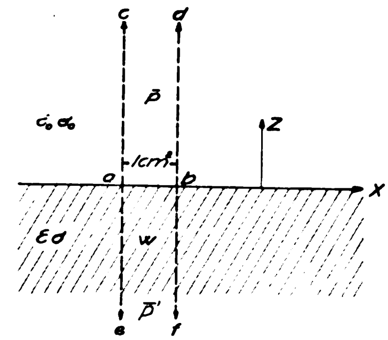

Fig. III. 5. p indicates the Field Energy in the Cylinder *acdb*, and w the Loss of Energy per second in the Cylinder *aefb*.

Supposing the intensity of the electric field at a certain point to have two harmonic components at right angles to each other, namely $E_1 = E_{1m}\cos\omega t$ and $E_2 = E_{2m}\cos(\omega t + \varphi)$, then the heat loss w will be determined by

$$w = \frac{1}{2}\sigma(E_{1m}^2 + E_{2m}^2) \quad [\text{e. m. u.; erg, cm}^{-3}, \text{ sec}^{-1}]. \tag{3}$$

In an infinitely long cylinder with a cross-sectional area of 1 cm², the axis of which is perpendicular to the boundary surface (see Fig. 5), the loss of electrical energy in the lower medium (earth) will consequently be

$$w = \frac{\sigma}{2}\int_0^\infty (|E_x|^2 + |E_z|^2)_{z=0}\, e^{-2\gamma z} dz = \frac{1}{4}\frac{\sigma}{\gamma}(|E_x|^2 + |E_z|^2)_{z=0}, \quad [\text{e. m u.; erg sec}^{-1}] \tag{4}$$

while in the upper medium (air) the electro-magnetic energy $\bar{p}$ inside the same cylinder is determined by

$$\begin{aligned} \bar{p} &= \frac{1}{8\pi}\int_0^\infty (|E_{0x}|^2 + |E_{0z}|^2)_{z=0} \cdot e^{-2\gamma_0 z} dz = \frac{1}{8\pi}\frac{1}{2\gamma_0}(|E_{0x}|^2 + |E_{0z}|^2)_{z=0} \quad [\text{e. s. u.; erg}] \\ &= \frac{1}{8\pi}\frac{1}{2\gamma_0 c^2}(|E_{0x}|^2 + |E_{0z}|^2)_{z=0} \quad [\text{e. m. u; erg}] \end{aligned} \tag{5}$$

the dielectric constant ε_0 of the atmosphere being assumed equal to unity.

A comparison between (4) and (5) shows that

$$w = 4\pi\sigma c^2 \frac{\gamma_0}{\gamma}\left(\frac{|E_x|^2 + |E_z|^2}{|E_{0x}|^2 + |E_{0z}|^2}\right)_{z=0} \cdot \bar{p}. \tag{6}$$

Considering a section of the electro-magnetic wave bounded by the planes $y = 0$ and $y = l$, we have then for this section a total loss of energy W per cm in the direction of propagation, which is

$$W = lw. \quad (\text{erg sec}^{-1}) \tag{7}$$

Similarly, the total electro-magnetic energy P inside the space bounded by the planes $y = 0$, $y = l$, $x = x_0$ and $x = x_0 + 1$ is given by

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$$P = l\bar{p} + l\bar{p}', \qquad \text{(erg)} \qquad (8)$$

where $\bar{p}'$ is the electro-magnetic energy inside the lower half of the cylinder (earth), see Fig. 5. We shall restrict ourselves, however, to a consideration of the conditions in the case of a not too poorly conducting soil, and in that case $\bar{p}'$ is negligible as compared to $\bar{p}$, so that we may safely write

$$P = l\bar{p} \qquad \text{(erg)} \qquad (9)$$

If the velocity of propagation of the waves is c', the rate of decrease of the energy will be determined by

$$dP = -W dt = -W \frac{dx}{c'},$$

or

$$d\bar{p} = -\frac{\bar{w}}{c'} dx,$$

or

$$\frac{d\bar{p}}{\bar{p}} = -\frac{\bar{w}}{c'\bar{p}} dx = -4\pi\sigma \frac{c^2\gamma_0}{c'\gamma} \left(\frac{|E_x|^2 + |E_z|^2}{|E_{0x}|^2 + |E_{0z}|^2} \right)_{z=0} dx. \qquad (10)$$

Hence

$$\bar{p} = \bar{p}_0 e^{-2\alpha' x}, \qquad (11)$$

where

$$\left.\begin{aligned} \alpha' &= \frac{\bar{w}}{2c'\bar{p}} = 2\pi\sigma \frac{c^2\gamma_0}{c'\gamma} \cdot \left(\frac{|E_x|^2 + |E_z|^2}{|E_{0x}|^2 + |E_{0z}|^2} \right)_{z=0} \\ &= 2\pi\sigma \frac{c^2\gamma_0}{c'\gamma} \sqrt{\frac{\left(\frac{\sigma_0}{\sigma}\right)^2 + \left(\frac{\omega}{\sigma}\frac{\varepsilon_0}{4\pi c^2}\right)^2}{1 + \left(\frac{\omega}{\sigma}\frac{\varepsilon}{4\pi c^2}\right)}}. \end{aligned}\right\} \text{(cm}^{-1}\text{)} \qquad (12)$$

Assuming here $\frac{\omega}{\sigma}\frac{\varepsilon}{4\pi c^2} \ll 1$ and $\varepsilon_0 = 1$, we have $c' = c$, and thus by means of the approximate formulas in Table 1 we obtain the obvious result $\alpha' = \alpha$.

We have made this deduction of the value of α on the basis of energy considerations, partly because we are going to use similar considerations in what follows and partly in order to emphasize how the attenuation of the wave is influenced by the total field energy per unit length in the direction of propagation. That the attenuation in the case considered assumes such exceedingly small values for the low frequencies is not so much because the loss of energy at these low frequencies is so very small, but far more because the total field energy of the wave motion concerned is so exceedingly large for the long waves. That such is the case will be seen immediately, when the expressions for $\bar{w}$ and $\bar{p}$ are modified so as to allow a direct comparison. We assume

$$\omega \ll \sigma \frac{4\pi c^2}{\varepsilon},$$

and have then

$$\bar{w} = \frac{1}{16\pi c^2}\sqrt{\frac{\omega}{2\pi\sigma}} \left(|E_{0x}|^2 + |E_{0z}|^2\right)_{z=0} = \frac{1}{16\pi c^{\frac{3}{2}}} \frac{1}{\sqrt{\sigma\cdot\lambda}} \left(|E_{0x}|^2 + |E_{0z}|^2\right)_{z=0}, \qquad (13)$$

and

$$\bar{p} = \frac{1}{16\pi}\frac{1}{\omega}\sqrt{\frac{8\pi\sigma}{\omega}} \left(|E_{0x}|^2 + |E_{0z}|^2\right)_{z=0} = \frac{\lambda\sqrt{\sigma\lambda}}{16\pi^2 c^{\frac{3}{2}}} \left(|E_{0x}|^2 + |E_{0z}|^2\right)_{z=0}. \qquad (14)$$

With short waves, however, the case is quite the opposite. Here it is especially the low field energy that causes the attenuation to become so large.

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3. *Propagation of Waves from a Dipole-Oscillator along a Plane Conducting Surface.*

The solution given by *H. Hertz* for the radiation from a dipole-oscillator in free space may be used without difficulty for the determination of the propagation of waves from a dipole vertically arranged on a plane conducting surface separating two media, one of which is a perfect insulator and the other a perfect conductor, and the result obtained will be the same as given in the Introduction, in formula (1). But the problem becomes far more intricate if we assume the conducting medium to be of finite conductivity and to have a finite dielectric constant, and in this form it was first solved by *A. Sommerfeld*[1], while *P. Epstein*[2] has illustrated this solution of the problem by diagrams showing the form of the lines of electric force.

In order to characterize the solution, *Sommerfeld* introduces the *numerical distance* ϱ which with the symbols used above is determined by

$$\varrho = \sqrt{\pi\omega}\,\frac{\left|\left(\sigma_0 - j\omega\frac{\varepsilon_0}{4\pi c^2}\right)^{\frac{3}{2}}\right| \cdot \left|\sigma_0 - \sigma + j\omega\frac{\varepsilon - \varepsilon_0}{4\pi c^2}\right|}{\sigma^2 + \omega^2\frac{\varepsilon^2}{16\pi^2 c^4}} \cdot r, \tag{15}$$

where r is the distance from the transmitter to the point considered, measured in cm.

Assuming

$$\sigma_0 = 0 \quad \text{and} \quad \varepsilon_0 = 1$$

equation (15) is reduced to

$$\varrho = \frac{\sqrt{\sigma^2 + \omega^2\frac{(\varepsilon - 1)^2}{16\pi^2 c^4}}}{\sigma^2 + \omega^2\frac{\varepsilon^2}{16\pi^2 c^4}} \cdot \frac{\omega^2}{8\pi c^3} \cdot r. \tag{16}$$

For $\frac{\omega}{\sigma}\frac{\varepsilon}{4\pi c^2} \ll 1$, corresponding to $2c\sigma\lambda \gg \varepsilon$ (λ in cm), we have approximately

$$\varrho = \frac{\omega^2}{8\pi\sigma c^3} \cdot r = \frac{\pi}{2c\sigma\lambda} \cdot \frac{r}{\lambda}, \quad [\text{r and } \lambda \text{ in cm}] \tag{17}$$

while for $\frac{\omega}{\sigma}\frac{\varepsilon - 1}{4\pi c^2} \gg 1$, corresponding to $2c\sigma\lambda \ll \varepsilon - 1$, we have

$$\varrho = \frac{\omega}{2c}\frac{\varepsilon - 1}{\varepsilon^2} \cdot r = \pi\frac{\varepsilon - 1}{\varepsilon^2}\frac{r}{\lambda}. \tag{18}$$

Sommerfeld has now shown that for a numerical distance $\varrho < \frac{1}{4}$, the propagation of the waves will go on very nearly in the same manner as if the earth was a perfect conductor, *i. e.* in accordance with *Hertz'* formula, (see the Introduction formula (1)). In this case the dispersion of the energy over a cross-sectional area increasing with the square of the distance is the principal factor determining the decreasing amplitude of the waves while the loss due to propagation along the plane surface is of secondary importance. We call this the spherical manner of propagation.

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[1] *A. Sommerfeld:* (a) Ann. der Phys. (4). Bd. 28, p. 665—736. 1909; Bd. 81, p. 1135—53. 1926. (b) Jahrb. d. drahtl. Tel. Bd. 4, p. 157—176. 1910.

[2] *P. Epstein:* Jahrb. d. drahtl. Tel. Bd. 4, p. 176—187. 1910.

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For large values of the numerical distance, however, *i. e.* for $\varrho \gg 1$, the main portion of the field will correspond to an annular or cylindrical propagation of the waves along the plane surface of the conductor, and *Sommerfeld* has shown that for this portion of the field the relation of the field intensity to the distance is determined by

$$\frac{E_r}{E_{r_0}} = \sqrt{\frac{r_0}{r}}\, e^{-\alpha(r-r_0)} \tag{19}$$

α has here the same value as the coefficient of attenuation determined by *Zenneck* (see formula (1)), while the factor $r^{-\frac{1}{2}}$ is due to the dispersion of energy caused by the cylindrical manner of propagation. For very high values of r the surface waves of *Sommerfeld* and *Zenneck* must necessarily behave in the same manner, and the attenuation therefore must obviously have the same value in the two cases. Consequently, the amplitude in the two media must also decrease with increasing distance from the boundary plane in the same manner in the two cases.

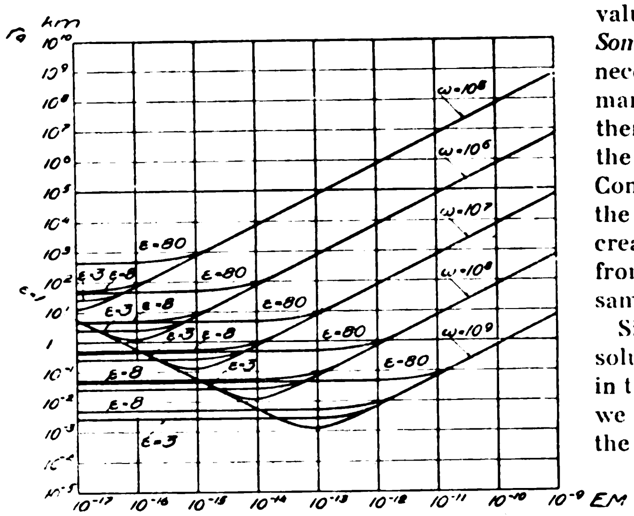

Fig. III. 6. The Critical distance r_g Corresponding to the Numerical Distance $\varrho = 1$ in *Sommerfeld*'s Theory.

Since the character of the solution is thus quite different in the two cases $\varrho \ll 1$ and $\varrho \gg 1$, we have in Fig. 6 represented the critical distance r_g corresponding to the numerical distance $\varrho = 1$, in the frequency range $\omega = 10^5$ to $\omega = 10^9$. For distances essentially shorter than r_g the propagation of the waves will be mainly spherical, while for distances essentially greater than r_g it will be mainly cylindrical.

An inspection of Fig. 6 shows that for $\omega > 3 \cdot 10^7$ and for $\sigma \leq 10^{-11}$ (*i. e.* for short waves in general) the critical distance r_g is so small that the numerical distance becomes much greater than unity for all the ranges of any importance in practice. For short waves we may therefore always assume a cylindrical manner of propagation.

On the other hand for ocean water ($\sigma \geq 10^{-11}$) and for $\omega < 10^7$ the numerical distance is generally considerably shorter than unity, and the propagation is thus spherical here.

Although *Sommerfeld*'s theory undoubtedly is more rational than *Zenneck*'s, and also agrees better with experience for the propagation of long waves over good conducting ground it cannot any more than the *Zenneck* theory give a satisfactory explanation of the manner of propagation of radio waves.

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4. *Propagation of Plane Waves Between Two Plane Parallel Conducting Surfaces.*

Let the distance between the plane boundary surfaces be equal to h, see Fig. 7, and let the system of co-ordinates be as indicated in the figure, then we have for the motion of a plane wave, in the direction of the X-axis: $E_y = 0$, $H_x = H_z = 0$ and $\frac{\partial}{\partial y} = 0$. The fundamental equations (I) and (II) will then be reduced to

$$-\frac{\partial H_y}{\partial t} = \frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x}, \tag{I'}$$

and

$$\left.\begin{aligned} 4\pi\sigma E_x + \frac{\varepsilon}{c^2}\frac{\partial E_x}{\partial t} &= -\frac{\partial H_y}{\partial z}, \\ 4\pi\sigma E_z + \frac{\varepsilon}{c^2}\frac{\partial E_z}{\partial t} &= \frac{\partial H_y}{\partial x}. \end{aligned}\right\} \tag{II'}$$

Inserting here

$$E_x = \mathbf{E}_x \cdot F,\ E_z = \mathbf{E}_z \cdot F,\ H_y = \mathbf{H}_y \cdot F$$
$$\text{and } F = e^{-sx - j\omega t},$$

where $\mathbf{E}_x$, $\mathbf{E}_z$, $\mathbf{H}_y$ and s are supposed to be independent of x, we get for (I') and (II'):

$$\left.\begin{aligned} -j\omega \mathbf{H}_y &= \frac{\partial \mathbf{E}_x}{\partial z} + s\mathbf{E}_z, \\ \left(4\pi\sigma + j\omega\frac{\varepsilon}{c^2}\right)\mathbf{E}_x &= -\frac{\partial \mathbf{H}_y}{\partial z}, \\ \left(4\pi\sigma + j\omega\frac{\varepsilon}{c^2}\right)\mathbf{E}_z &= -s\mathbf{H}_y. \end{aligned}\right\} \tag{III}$$

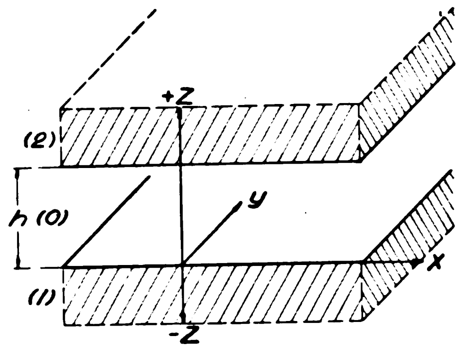

Fig. III. 7.
Symbols and System of Co-ordinates.

In accordance with Fig. 7 we shall indicate all symbols concerning the intermediate medium (air) with the index 0, while the lower medium is characterized by the index 1 and the upper medium by the index 2.

We assume a set of solutions of the following form[1]:

$$\left.\begin{aligned} \mathbf{H}_{0y} &= a_0 e^{r_0 z} + b_0 e^{-r_0 z}, \\ \mathbf{E}_{0x} &= -\frac{r_0}{4\pi\sigma_0 + j\omega\frac{\varepsilon_0}{c^2}}\left(a_0 e^{r_0 z} - b_0 e^{-r_0 z}\right), \\ \mathbf{E}_{0z} &= -\frac{s}{4\pi\sigma_0 + j\omega\frac{\varepsilon_0}{c^2}}\left(a_0 e^{r_0 z} + b_0 e^{-r_0 z}\right), \end{aligned}\right\} \tag{20}$$

$$\left.\begin{aligned} \mathbf{H}_{1y} &= a_1 e^{r_1 z}, \\ \mathbf{E}_{1x} &= -\frac{r_1}{4\pi\sigma_1 + j\omega\frac{\varepsilon_1}{c^2}} a_1 e^{r_1 z}, \\ \mathbf{E}_{1z} &= -\frac{s}{4\pi\sigma_1 + j\omega\frac{\varepsilon_1}{c^2}} a_1 e^{r_1 z}, \end{aligned}\right\} \tag{21}$$

[1] *F. Hack:* Ann. d. Physik. (IV). Bd. 27. p. 43—63. 1908.

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and
$$\left.\begin{aligned}
H_{2y} &= a_2 e^{r_2 z},\\
E_{2x} &= -\frac{r_2}{4\pi\sigma_2 + j\omega\frac{\varepsilon_2}{c^2}}\, a_2 e^{r_2 z},\\
E_{2z} &= -\frac{s}{4\pi\sigma_2 + j\omega\frac{\varepsilon_2}{c^2}}\, a_2 e^{r_2 z},
\end{aligned}\right\} \tag{22}$$

which will satisfy the equations (III), provided that

$$\left.\begin{aligned}
r_0^2 + s^2 &= j\omega\left(4\pi\sigma_0 + j\omega\frac{\varepsilon_0}{c^2}\right) = p_0^2,\\
r_1^2 + s^2 &= j\omega\left(4\pi\sigma_1 + j\omega\frac{\varepsilon_1}{c^2}\right) = p_1^2,\\
r_2^2 + s^2 &= j\omega\left(4\pi\sigma_2 + j\omega\frac{\varepsilon_2}{c^2}\right) = p_2^2.
\end{aligned}\right\} \tag{23}$$

Further, the following boundary conditions have to be satisfied:

$$\left.\begin{aligned}
&\text{For } z = 0 \text{ we must have } H_{0y} = H_{1y} \text{ and } E_{0x} = E_{1x}\\
&\text{and for } z = h \text{ we must have } H_{0y} = H_{2y} \text{ and } E_{0x} = E_{2x}.
\end{aligned}\right\} \tag{24}$$

Hence

$$\left.\begin{aligned}
&a_1 = a_0 + b_0, && \text{(a)}\\
&a_2 e^{r_2 h} = a_0 e^{r_0 h} + b_0 e^{-r_0 h}, && \text{(b)}\\
&\frac{r_0}{p_0^2}(a_0 - b_0) = \frac{r_1}{p_1^2}(a_0 + b_0), && \text{(c)}\\
&\frac{r_0}{p_0^2}\left(a_0 e^{r_0 h} - b_0 e^{-r_0 h}\right) = \frac{r_2}{p_2^2}\left(a_0 e^{r_0 h} + b_0 e^{-r_0 h}\right). && \text{(d)}
\end{aligned}\right\} \tag{25}$$

From equation (25c) follows

$$b_0 = \frac{r_0 p_1^2 - r_1 p_0^2}{r_0 p_1^2 + r_1 p_0^2}\, a_0, \tag{26}$$

and the last equation (25d) then gives

$$e^{2r_0 h} = \frac{r_0 p_1^2 - r_1 p_0^2}{r_0 p_1^2 + r_1 p_0^2} \cdot \frac{r_0 p_2^2 + r_2 p_0^2}{r_0 p_2^2 - r_2 p_0^2}. \tag{27}$$

Equation (27) in connection with equations (23) serves to determine the four parameters s, r_0, r_1 and r_2. Subsequently b_0 is determined from equation (26) and a_1 and a_2 from equations (25a) and (25b). The formulas (20) to (22) then give the desired solution, since all the amplitudes are proportional to a_0.

We shall now consider some special cases which also furnish a check on the correctness of the formulas deduced above:

a) The two media (0) and (2) are supposed to be alike. Then $p_2 = p_0$ and $r_2 = r_0$; consequently $r_0 p_2^2 - r_2 p_0^2 = 0$, so that equation (27) is satisfied only when $r_0 p_1^2 - r_1 p_0^2 = 0$, which gives $b_0 = 0$, but equation (27) is then satisfied for all values of h and the equation $r_0 p_1^2 = r_1 p_0^2$ in connection with the first two of equations (23) are just the same as those serving to determine the parameters s_1, r_0 and r in the case treated by *Zenneck*, (see formulas (1')), which case, under the assumptions made, is identical to the one here considered.

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b) Supposing $h = \infty$, then $e^{2r_0h} = 0$, and the equation (27) will be satisfied by $r_0 p_1^2 - r_1 p_0^2 = 0$, corresponding to $b_0 = 0$. The case is identical to the one treated above.

c) Supposing the two media (1) and (2) to be alike, resulting in $p_2 = p_1 = p$ and, consequently, $r_2 = -r_1 = r$, then (27) becomes

$$e^{r_0 h} = \frac{r_0 p^2 + r p_0^2}{r_0 p^2 - r p_0^2}, \tag{28}$$

and corresponding to this

$$b_0 = \frac{r_0 p^2 + r p_0^2}{r_0 p^2 - r p_0^2} a_0 = e^{r_0 h} a_0. \tag{29}$$

Inserting this value in equation (20) we have:

$$\left.\begin{aligned}
H_{0y} &= 2a_0 e^{\frac{1}{2} r_0 h} \cosh [r_0 (z - \tfrac{1}{2} h)],\\
E_{0x} &= -\frac{r_0}{4\pi\sigma_0 + j\omega \frac{\varepsilon_0}{c^2}} 2a_0 e^{\frac{1}{2} r_0 h} \sinh [r_0 (z - \tfrac{1}{2} h)],\\
E_{0z} &= -\frac{s}{4\pi\sigma_0 + j\omega \frac{\varepsilon_0}{c^2}} 2a_0 e^{\frac{1}{2} r_0 h} \cosh [r_0 (z - \tfrac{1}{2} h)].
\end{aligned}\right\} \tag{30}$$

The equations (30) show the electric and the magnetic fields in the intermediate medium to be symmetrical with respect to the central plane $n'n$ of the latter, see Fig. 8 I. The line $m'nm''$ indicates an electric line of force.

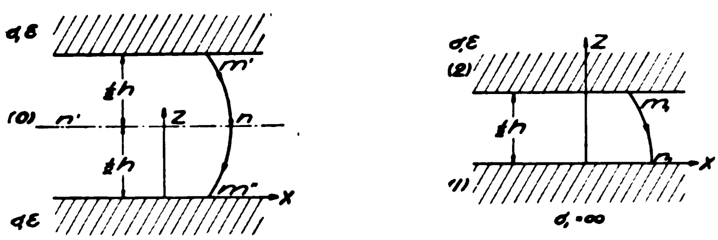

Fig. I. Fig. II.

Fig. III. 8. The Attenuation is of the same Value in I and II.

d) Assuming $\sigma_1 = \infty$ and consequently also $p_1 = \infty$, we get from equation (27)

$$e^{2r_0 h} = \frac{r_0 p_2^2 + r_2 p_0^2}{r_0 p_2^2 - r_2 p_0^2}, \tag{31}$$

while equation (26) shows that

$$b_0 = a_0. \tag{32}$$

Inserting this value of b_0 in equation (20) we find:

$$\left.\begin{aligned}
H_{0y} &= 2a_0 \cosh (r_0 z),\\
E_{0x} &= -\frac{r_0}{4\pi\sigma_0 + j\omega \frac{\varepsilon_0}{c^2}} \cdot 2a_0 \sinh (r_0 z),\\
E_{0z} &= -\frac{s}{4\pi\sigma_0 + j\omega \frac{\varepsilon_0}{c^2}} \cdot 2a_0 \cosh (r_0 z).
\end{aligned}\right\} \tag{33}$$

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A comparison between the equations (33) and (30) as well as between equations (31) and (28) shows that when σ_2 and ε_2 in this case are chosen equal to σ and ε in case c), and when at the same time the height h of the medium (0) is taken half as large as in case c), then the field in medium (0) will be the same in the two cases, as indicated in Fig. 8 II. The interesting point here is that the attenuation α becomes the same in the two cases. This result is also immediately evident by a direct comparison between Figs. 8 I and 8 II.

e) Assuming, finally, $|\gamma_0 h| \gg 1$, where γ_0 is negative and forms the real part of r_0, then e^{2r_0h} will be very nearly equal to zero, and equation (27) will be satisfied when either

$$r_0 p_1^2 - r_1 p_0^2 = 0, \tag{34}$$

or

$$r_0 p_2^2 + r_2 p_0^2 = 0. \tag{35}$$

These equations are exactly the same as those obtained in the case treated by *Zenneck* with only one boundary surface, (see formulas (1')), since (35) corresponds to a *Zenneck* wave along the boundary surface between the media (0) and (2). The fields of the two waves will overlap only to an imperceptible small degree, and the two waves will move independently of each other.

A general determination of the attenuation constants α, γ_0, γ_1 and γ_2 on the basis of formulas (23) and (27) and a graphical representation of the values of the said coefficients as functions of σ_1, ε_1, σ_2, ε_2, h and ω offers no principal difficulties; but as the work of computation would be quite considerable, and the representation by curves would require too much space, we shall in the following consider only a few examples of the simpler cases mentioned above under headings c) and d).

We therefore now pass to the case where $\sigma_1 = \sigma_2 = \sigma$ and $\varepsilon_1 = \varepsilon_2 = \varepsilon$. The equation (28) may then be written in the somewhat simpler form:

$$\mathrm{tgh}\,\frac{r_0 h}{2} = \frac{r p_0^2}{r_0 p^2}, \tag{36}$$

and this equation in connection with the equations

$$\left.\begin{aligned} r_0^2 + s^2 &= (\gamma_0 + j\delta_0)^2 + (\alpha + j\beta)^2 = p_0^2, \\ r^2 + s^2 &= (\gamma + j\delta)^2 + (\alpha + j\beta)^2 = p^2. \end{aligned}\right\} \tag{23'}$$

and

serves to determine the parameters s, r_0 and r.

In the following we assume the intermediate medium to be non-conducting, corresponding to $\sigma_0 = 0$. In that case we have:

$$p_0^2 = -\varepsilon_0 \frac{\omega^2}{c^2} = -\frac{4\pi^2}{\lambda^2}. \qquad (\lambda \text{ in cm}) \tag{37}$$

We make the further assumption that $\frac{\omega}{\sigma}\frac{\varepsilon}{4\pi c^2} \ll 1$ or $\frac{\varepsilon}{2c\sigma\lambda} \ll 1$ (λ in cm), which gives us

$$p^2 = j\,4\pi\sigma\omega, \tag{38}$$

and equation (36) then becomes

$$\mathrm{tgh}\,\frac{r_0 h}{2} = j\,\frac{r}{r_0}\frac{\varepsilon_0\omega}{4\pi c^2\sigma}. \tag{36'}$$

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With the problem thus simplified, we shall, however, consider only the simple limiting case, where the height h of the intermediate medium is so small that the field intensity has about the same value throughout. We assume therefore $|r_0 h| \ll 1$, so that equation (36') may be written approximately:

$$\frac{r_0 h}{2} = j \frac{r}{r_0} \frac{\varepsilon_0 \omega}{4\pi c^2 \sigma} \quad \text{or} \quad r_0^4 = -\frac{\varepsilon_0^2 \omega^2}{4\pi^2 c^4 \sigma^2 h^2} r^2. \tag{39}$$

Elimination of r_0 and r between (39) and (23') results in:

$$(p_0^2 - s^2)^2 + \frac{\varepsilon_0^2 \omega^2}{4\pi^2 \sigma^2 c^4 h^2}(p_0^2 - s^2) + \frac{\varepsilon_0^2 \omega^2}{4\pi^2 \sigma^2 c^4 h^2}(p^2 - p_0^2) = 0. \tag{40}$$

Under the above mentioned assumption that $\frac{\omega}{\sigma} \cdot \frac{\varepsilon_0}{4\pi c^2} \ll 1$, and provided that the height of the intermediate medium is equal to or exceeds one km, *i. e.* $h \geqq 10^5$ cm, we get from equation (40) the following approximate formula:

$$s^2 = -\varepsilon_0 \frac{\omega^2}{c^2}\left(1 - j \frac{1}{h\sqrt{2\pi\sigma\omega}}\right),$$

or

$$s = \alpha + j\beta = j\sqrt{\varepsilon_0} \frac{\omega}{c}\left(1 - j \frac{1}{2h\sqrt{2\pi\sigma\omega}}\right), \tag{41}$$

whence

$$\alpha = \frac{1}{2ch}\sqrt{\frac{\varepsilon_0 \omega}{2\pi\sigma}} \quad \text{and} \quad \beta = \frac{\omega}{c}\sqrt{\varepsilon_0}. \tag{42}$$

The corresponding values of r_0 and r are furnished by (23'), and are determined by

$$\gamma_0 = -\delta_0 = \frac{\omega^{\frac{3}{4}}\sqrt{\varepsilon_0}}{c\sqrt{2h}\sqrt[4]{2\pi\sigma}} \quad \text{and} \quad \gamma = \delta = \sqrt{2\pi\omega\sigma}. \tag{43}$$

A comparison with the formulas in Table 1 shows that the values found for γ and δ are the same as in case of wave propagation along one single boundary surface. The penetration of the waves into the conducting media consequently takes place in the same manner in the two cases. The parameter β also having the same value, the velocity of the waves will be the same in both cases.

According to the assumptions made, the field intensity is very nearly of the same value throughout the entire height of the intermediate medium. Consequently the approximate value of the field energy p in the medium (o) inside a cylinder with a cross-sectional area equal to unity and of height h cm may be written (see formula (5)):

$$\bar{p} = \frac{\varepsilon_0}{8\pi} \frac{1}{c^2} h\left(|E_{0x}|^2 + |E_{0z}|^2\right)_{z=0}, \quad \text{[e. m. u.; erg]} \tag{44}$$

while the loss w' of energy per cm² of the boundary surfaces and per second is determined by

$$w' = 2w = \frac{1}{2} \frac{\sigma}{\gamma}\left(|E_x|^2 + |E_z|^2\right)_{z=0} \quad \text{[e. m. u.; erg sec}^{-1}\text{]}. \tag{45}$$

According to equation (12) the attenuation α' is therefore determined by

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$$\alpha' = \frac{w'}{2c'p} = \frac{w'\sqrt{\varepsilon_0}}{2cp} = \frac{1}{2ch}\sqrt{\frac{\varepsilon_0 \omega}{2\pi\sigma}} = \alpha. \tag{46}$$

The energy calculations performed above, in equations (12), thus lead to the same values of the attenuation constant as equation (42).

If we had assumed $\sigma_1 = \sigma$ and $\sigma_2 = \infty$ instead of $\sigma_1 = \sigma_2 = \sigma$ the value of the attenuation would obviously come out only half of that determined by (42) and (46), namely

$$\alpha = \frac{1}{4ch}\sqrt{\frac{\varepsilon_0 \omega}{2\pi\sigma}}. \tag{47}$$

Although it would be quite possible, without any serious difficulty, to extend somewhat further the exact treatment of the problem in question, we shall nevertheless abandon this course, and merely utilize the results gained for obtaining an approximately correct solution of a somewhat similar problem which presumably is of more practical importance than the one treated above.

We have so far considered only the propagation of a single plane wave in the space between the two plane conductors (1) and (2), whereby conditions adjust themselves so that the field-intensity varies throughout the intermediate layer according to some definite law, the maximum values being reached at the boundary surfaces. As shown above, the field strength may under certain conditions be nearly constant throughout the entire height of the layer, namely in case of good conductors and not too great height of the layer or, passing through all the intermediate stages, the other limiting case may be reached where two mutually independent waves are propagated one along each one of the boundary surfaces, while somewhere in the interior of the layer the intensity of field is very nearly equal to zero. The last mentioned case occurs when the height of the layer is large and the conductivity of the conductors is low.

In practice the conditions influencing the propagation of radio waves are of such a nature that even if the assumption of propagation between two surfaces be justified, these surfaces are at any rate not plane, but mainly spherical having for their centre the centre of the earth. Nor are these surfaces perfectly smooth but more or less irregular. Nor, finally, is the intermediate layer (the atmosphere) perfectly homogeneous, but its dielectric constant, and perhaps also its conductivity, is or may be somewhat varying, and quite pronounced surfaces of discontinuity may exist in the atmosphere as will be seen later. The result of all this will be a natural tendency for the mean field intensity, and consequently also for the mean field energy, to assume the same value throughout the entire height of the intermediate layer. In this manner, and therefore also in practice, the wave motion becomes far more complicated than assumed above. But on the other hand the question of primary interest to practice will not so much be an exact determination of the value of the field-intensity for some definite locality at some definite moment, because the field-intensity will generally depend on the local features of the place and, besides, will vary greatly from time to time. Of much more interest would be the determination of an attenuation constant useful in practice for estimating the range of the waves in relation to the mean value of the attenuation for any distance, for the wavelength used and for any hour of the day or night. It is reasonable to assume that the mean

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value of the effective field intensity taken throughout the entire height of the intermediate layer gives a more rational basis for determination of the attenuation than the value of the field intensity at one single definite point and at some definite moment.

We are thus facing the problem of determining how the field energy decreases when the waves are propagated substantially parallel to the boundary surfaces, while on the other hand irregularities present in the intermediate medium (air) and in the dividing surfaces may be so large that the field-intensity will have substantially the same value throughout the height of said layer. It is obviously impossible to present any exact solution of this problem, but we may obtain a useful approximation by assuming that the loss of energy occurring at the conducting surface of the earth is of the same value as the loss of energy which would be caused by a plane *Zenneck* wave propagating along the boundary surface (0)—(1), provided the field-intensity for this wave is equal to the mean value of the field-intensity in medium (0).

With regard to the loss of energy in the outer conducting shell the conditions are somewhat more intricate, and we cannot enter into this question until later. If we temporarily assume that the loss in the outer shell taken by itself (that is in case of a perfectly conducting earth) would give an attenuation constant $\alpha(\sigma_2)$, then the resultant attenuation $\alpha(\sigma_1, \sigma_2)$ may be written

$$\alpha(\sigma_1, \sigma_2) = \alpha(\sigma_1) + \alpha(\sigma_2), \tag{48}$$

where $\alpha(\sigma_1) = \alpha(\sigma_1, \infty)$ is the attenuation for the case $\sigma_2 = \infty$.

Assuming $\frac{\omega}{\sigma_1}\frac{\varepsilon}{4\pi c^2} \ll 1$ we get from the equations (47) and (48):

$$\alpha(\sigma_1, \sigma_2) = \frac{1}{4ch}\sqrt{\frac{\varepsilon_0\omega}{2\pi\sigma_1}} + \alpha(\sigma_2). \tag{49}$$ [1]

While, for $\frac{\omega}{\sigma_1}\frac{\varepsilon}{4\pi c^2} \gg 1$, we similarly find (Table 1 and equation (12)) that

$$\alpha(\sigma_1, \sigma_2) = \frac{1}{2h}\frac{\varepsilon_0 + \varepsilon_1}{2\,\varepsilon_0 + \varepsilon_1}\sqrt{\frac{\varepsilon_0}{\varepsilon_1}} + \alpha(\sigma_2). \tag{50}$$

In both these calculations of $\alpha(\sigma_1, \sigma_2)$, we have for the determination of the value of $\bar{p}$, — see equations (8) and (9) — considered only that portion of the field energy which is present in the medium (0) alone, although when using the formula (12) in the value of $\bar{p}$ we ought to have included the total field energy in all three media. The corresponding correction, however, is very small as long as σ_1 and σ_2 are not very small, and as long as the height of the layer is not very small, for instance not less than 10 km. We shall therefore not enter further into the calculation of the size of this correction, but merely note that owing to the said correction the lines in Fig. 9 corresponding

[1] The values of the attenuation constant in the case of wave propagation between two spherical boundary surfaces have been found by *G. N. Watson* (Proc. Roy. Soc. (A) Vol. 95, p. 546—563, 1919) and have been used by *Round*, *Eckersley*, *Tremellen* and *Lunnon* (Journ. I. E. E. Vol. 63, p. 994, 1925) in an important paper on the propagation of long waves. *Watson*'s values are $\sqrt{2}$ times as great as those given by the equations (42) and (49).

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to formula (50), which according to this formula ought to be horizontal, will really be curved upward to an extent increasing with decreasing values of the height h of the layer.

Figure 9 gives the value of $\frac{1}{\alpha(\sigma_1, \infty)}$ for the same range of frequencies and conductivities as those in Figs. 2 to 4. Later on there will be an opportunity for a more detailed discussion of the results thus obtained, which for long waves and for certain hours of the day and night will prove to give a mainly correct representation of the actual conditions. This representation, on the

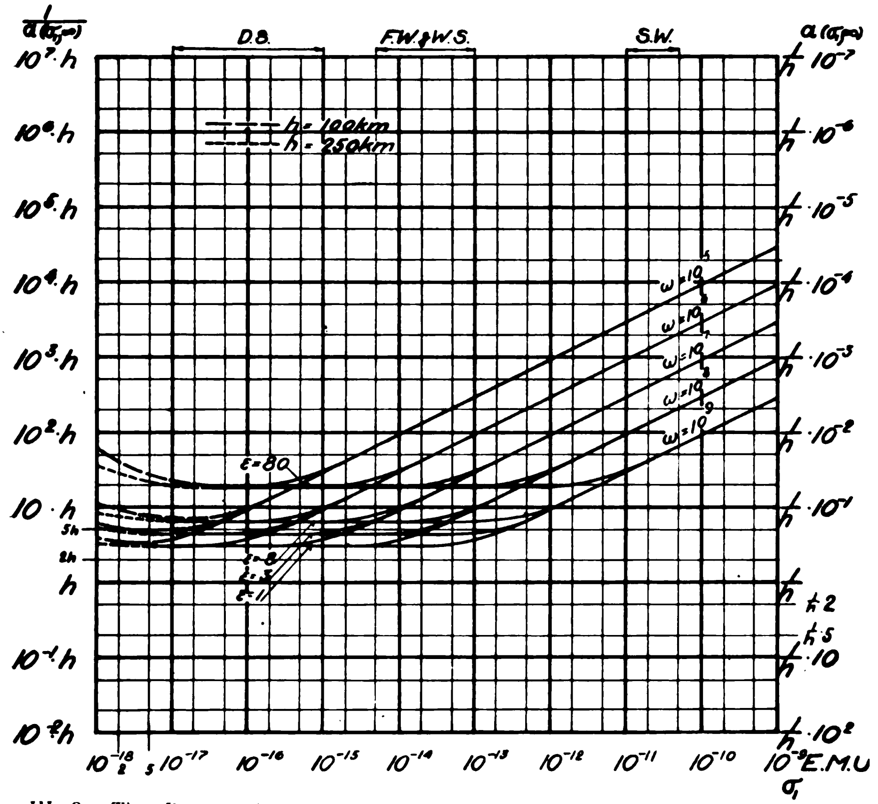

Fig. III. 9. The Range of the Waves, under the Assumption of Uniform Distribution of the Field Energy in the Air Space of h km between the Conductors σ_1 and $\sigma_2 = \infty$.

other hand, is unable to give any explanation either of the great reaching ability of short waves during favourable conditions, or of the great dependency of the transmission on the position of the sun relatively to the transmitter and receiver, — at any rate when a constant value of the height h is assumed.

CHAPTER IV.

THE COMPOSITION AND PRESSURE OF THE ATMOSPHERE AT GREAT HEIGHTS

1. *The Composition and Pressure of the Atmosphere.*

In the following chapters we shall be dealing with the question of the state of ionization of the atmosphere for various conditions and up to quite great heights above the earth's surface and also with the question of to what degree the ions and electrons contribute to the conductivity and dielectric constant

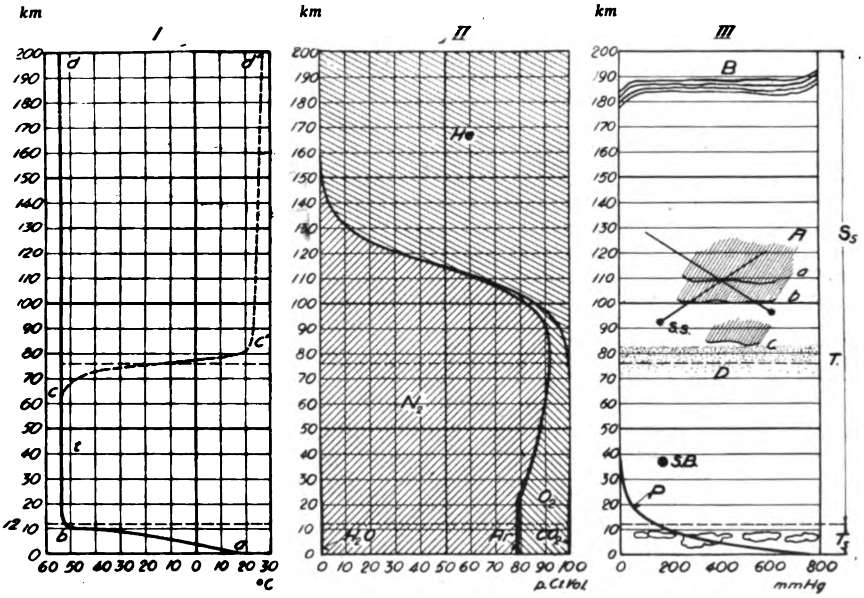

Fig. IV. 1. A Diagrammatic Representation of the State of the Atmosphere up to a Height of 200 km. In part I the curve *abcd* indicates the distribution of temperature assumed in this paper, while the curve *abcc'd'* indicates the distribution assumed by *Lindemann* and *Dobson*. Part II gives the percentage composition of the atmosphere marked F in Table 3 below and part III indicates the variation of the total air pressure *p* and gives the location of aurora borealis (*B*, *A*, *a*, *b*, *c*) and meteors. S. B. indicates the maximum height reached by sounding balloons.

of the atmosphere. However, since this requires a certain knowledge of the density and composition of the atmosphere from the earth's surface and up through the entire layer of air influencing the propagation of radio waves we have found it necessary here to investigate first the question of the composition and pressure of the atmosphere at various heights.

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In the lower portion of the atmosphere, the troposphere, extending from the surface of the earth and up to a height of about 12 km, conditions are known to vary. Ascending and descending currents exist here and cause masses of air from the various heights within this region to intermix so that the percentage composition of the air is mainly the same throughout the troposphere. This is also the region where clouds are formed, and where rain originates. The temperature inside the troposphere decreases more or less uniformly with increasing height reaching at the upper boundary a value of about -55° C. (See Fig. 1, where the curve *ab* represents the approximate average value of the temperature).

These conditions, however, are somewhat different for different parts of the globe. Generally speaking the boundary between the troposphere and the stratosphere, which boundary is frequently called the tropopause, is situated somewhat higher, about 12 to 15 km, in the neighbourhood of the equator, and somewhat lower, about 7 to 8 km, in the polar regions. Accordingly the temperature at the tropopause in the equatorial regions reaches values down to about -80°C or even less[1], and in the polar regions down to about -50° C. We shall assume here that the tropopause is situated at an average height of 12 km, that the temperature prevailing there has reached a value of about -55° C. and that this is also the temperature found in the entire stratosphere, when not otherwise expressly stated.

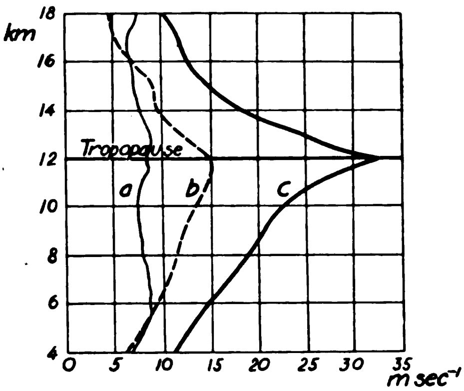

Fig. IV. 2. Mean Variation of Wind Velocity with Height. Data are grouped according to wind velocity in the highest two kilometres of the Troposphere: (a) less than 13 m per sec., (b) 13 to 19 m per sec., (c) greater than 19 m per sec.

Within the troposphere the mixture of air, water vapour only excepted, must be assumed to be fairly uniform, and even for the lower part of the stratosphere we must assume that some mixing of air takes place, on account of the considerable wind velocities which may be found up to an altitude of about 20 km, in spite of the fact that these velocities decrease rapidly upward from the tropopause[2]. Fig. 2 shows the wind distribution for heights between 4 and 18 km according to *Dobson*.

The Figure shows that strong winds have a pronounced maximum at the tropopause at a height of 12 km and then decrease rapidly with increasing heights. But it also appears that considerable wind velocities may still be found at a height of 18 km.

An effective mixing of air must therefore be assumed to extend partly up into the stratosphere. *Chapman* and *Milne*[3], for instance, give 20 km as the most probable value of the height of effective mixing (l. c. p. 359).

Higher up in the stratosphere there are no ascending and descending air

[1] See for instance: *W. van Bemmelen:* Koninklijk Magn. en Met. Observatorium te Batavia. Verh. No. 4 (Batavia 1916).

[2] *G. M. B. Dobson:* Q. J. R. Meteor. Soc. Vol. 46, p. 54—62. 1920.

[3] *S. Chapman* and *E. A. Milne:* Q. J. R. Meteor. Soc. Vol. 46, p. 357—397. 1920.

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currents, but the pressures of the individual gases in the air decrease upward as determined by the theory of gases.

Since the rapidity in the upward decrease of the pressure varies according to the density of the different gases, the composition of the air will also be different at different heights, the lighter gases becoming more prominent as the height increases.

We shall therefore next investigate the air pressure p as a function of the altitude h for a gas of density ϱ given by:

$$\varrho = \frac{\varrho_0}{p_0} \cdot \frac{p}{1+\alpha t} = \frac{1}{H_0} \cdot \frac{p}{1+\alpha t} = \frac{p}{H_t}, \quad \text{(p in g per cm}^2) \tag{1}$$

where p_0 is the pressure of the gas and ϱ_0 its density at the surface of the earth for $t = 0^0$ C., while

$$H_0 = \frac{p_0}{\varrho_0} \quad (p_0 \text{ in g per cm}^2,\ \varrho_0 \text{ in g per cm}^3,\ H_0 \text{ in cm}) \tag{2}$$

is the height required by a homogeneous layer of the gas of density ϱ_0 throughout in order to exert the same pressure p_0 at the surface of the earth. We indicate by H_0 the height of this »homogeneous atmosphere« of the gas concerned, and in Table 2 we have given for a number of gases the values of H_0 together with the densities ϱ^0 of the individual gases at 0^0 C. and 760 mm.

In (1) we have substituted

$$H_t = H_0(1 + \alpha t), \tag{3}$$

where $\alpha = \frac{1}{273}$.

The increase dp in pressure due to an increment of height dh, is determined by:

$$-dp = \varrho dh = \frac{1}{H_0} \cdot \frac{p}{1+\alpha t} \cdot dh = \frac{p}{H_t} \cdot dh, \tag{4}$$

or

$$\frac{dp}{p} = -\frac{dh}{H_0(1+\alpha t)} = -\frac{dh}{H_t},$$

consequently:

$$p = p_0 \cdot e^{-\frac{h}{H_0(1-\alpha t)}} = p_0 \cdot e^{-\frac{h}{H_t}}. \tag{5}$$

By using formulas (1) to (5), and on the basis of the values in Table 2, the partial pressure and density of the individual gases contained in the atmosphere may be computed, when the height of effective mixing has been fixed, and the temperatures at the various heights are known. The total pressure is then the sum of the partial pressures of the individual constituents.

It is still necessary to add to Table 2 some remarks regarding the composition of the atmosphere. There is no essential disagreement concerning the percentages of nitrogen, oxygen, carbon dioxide, argon, neon and helium[1–2] in the

[1] An exception is formed by a Paper by *G. J. Elias* (Jahrb. d. drahtl. Tel. Bd. 27 p. 66—73, 1926) who assumes the upper atmosphere to consist solely of nitrogen, and the temperature to be -53^0 C. For great heights he therefore arrives at exceedingly low pressures, so low that presumably they cannot be made to agree with the experiences concerning meteors, a point which we shall come back to later. These pressures are not given directly in the above mentioned paper, but they may be computed on the basis of the table on page 70, as mentioned in the following section.

[2] *L. Vegard*'s views form an exception, and we shall later revert to this point.

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Table 2. Constants for some of the Constituents of the Atmosphere.

Gas	H_2	He	H_2O	Ne	N_2	O_2	Ar	CO_2	Atm.[3]	
Height of the »Homogeneous Atmosphere«[1] at 0^0 C. .... $H_0 =$	115.0	58.42	12.83	11.60	8.26	7.23	5.80	5.23	8.00	km
Composition of the Atmosphere at the Surface of the Earth[2]	(0,0005)	0.0004	(1.2)	(0.0012)	78.08	20.95	0.94	0.03	100	per cent by volume
Partial Pressure of the Individual Gases[4]	(0.004)	0.00304		0.0091	593.41	159.22	7.144	0.228	760.0	mm Hg
Density at 0^0C. and $p=760$ mm. . $\varrho^0 =$	0.090	0.177	0.806	0,891	1.251	1.429	1.781	1.977	1.293	10^{-3} g·cm^{-3}
$(n-1)\cdot 10^6 = A\varrho^0\cdot 10^6$ .... = (for $\lambda = \infty$, $p = 760$ mm, $t = 15^0$ C.)	139 [5]	35 [6]	250 [7]	67 [8]	290 [9]	265 [9]	278 [11]	441 [9]	272 [9, 10]	
$R = \frac{H_0}{n-1}$ =	828000		51400		28500	27300	20900	11900	29400	km
$\frac{R}{R_0} = \frac{R}{6370}$ =	130		8.1		4.5	4.3	3.3	1.86	4.6	

[1] *A. Wegener:* Thermodynamik der Atmosphäre, p. 39. (Leipzig, 1911).
Hann und *Süring:* Lehrb d. Meteorologie. 4' Aufl. p. 8. (Leipzig 1926).

[2] *W. J. Humphreys:* Physics of the Air, p. 69. (Philadelphia, 1920). *Hann* u. *Süring:* l. c., p. 5. *S. Chapman* and *E. A. Milne:* l. c, p. 364.

[3] The figures apply to dry air.

[4] *Hann* u. *Süring:* l. c., p. 7. The values of partial pressures given for helium and neon do not agree with the volumetric percentages given by the same authors.

[5] *St. Loria:* Die Lichtbrechung in Gasen, p. 43. (Braunschweig 1914).

[6] » : l. c., p. 49.

[7] » : l. c., p. 59. Correct for $\lambda = 6708\cdot 10^{-8}$ cm, but not for the wave lengths used in radio work.

[8] » : l. c., p. 53.

[9] *E. Stoll:* Ann. d. Phys. (IV). Bd 69, p. 81. 1922

[10] The value holds good for red light.

[11] *B. Quarder:* Ann. d Phys. (IV) Bd. 74, p 255—274 1924.

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Table 3. Data for certain ›Atmospheres‹ to be discussed later.

No.	H	W	E [2]	A [1]	B [1]	C [1]	D	F and F' [4]	Units
Hydrogen Content in Lower Atmosphere	$1\cdot10^{-2}$	$3.3\cdot10^{-3}$	0	$5\cdot10^{-4}$	$5\cdot10^{-4}$	0	0	0	p. Ct. vol.
Height of Effective Mixing H_M ..	11			20	12	12	20	12	km
Pressure of Water Vapour at Height H_M	$1.7\cdot10^{-2}$			$3\cdot10^{-3}$	$1.3\cdot10^{-3}$	$1.3\cdot10^{-3}$	various	various	mm
Total Pressure at the Tropopause (12 km)				132.1	143.2	143.2	132.1	143.2	mm
Total Pressure at Height H_M ...	168.0			41.2	143.2	143.2	41.2	143.2	mm
Temperature in the Stratosphere	219	219	220	[3]	219	219	219	219	^{0}abs

1 Due to an error in *Hann* u. *Süring* (l. c. p. 7) the partial pressure of helium is taken as $p = 0.00114$ mm instead of $p = 0.00304$ and the partial pressure of neon as $p = 0.0114$ mm instead of $p = 0.0091$ mm.

2 It is assumed that the atmosphere at high altitudes only contains nitrogen.

3 Between 20 and 60 km it is assumed that the temperature is equal to $t = 219^0$ abs, while for altitudes higher than 60 km the temperature is equal to $t = 300^0$ abs.

4 The ›atmosphere‹ *F'* has the same composition, temperature and pressure as *F*, the only difference being in the assumed value of the mean free path of the electrons, see chapters IX and XI.

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lower atmosphere but in regard to water vapour and hydrogen the case is different. At -54^0 C. the saturated vapour pressure of ice is about 0.019 mm[1], and as the relative humidity is always found to be small, the actual pressure of water vapour will be much smaller than this, but it is impossible beforehand to fix any definite value. We are therefore postponing until later the treatment of the question of the quantity of water vapour in the outer atmosphere. As to the percentage of hydrogen contained in the lower atmosphere the data at hand vary very much: Thus *W. J Humphreys* (l. c., p. 69, 1920) gives 0.01 per cent by volume, while *Hann* and *Süring* (l. c., p. 5, 1926) give 0.001 per cent by volume. Lord *Rayleigh*[2] found that the hydrogen percentage in air is likely to be below 0.003 per cent by volume. *G. Claude*[3] arrived at the result that the percentage by volume of hydrogen must be about 0.0001, although he also found values far lower than this. *J. H. Jeans*[4], on the basis of the measurements available, gives 0.001 per cent by volume as a probable value, while *A. Krogh*[5] found the percentage by volume of hydrogen to be certainly below 0.0005 and probably below 0.0002. *Chapman* and *Milne* (l. c., p. 362–363) do not definitely answer the question, but say: »Having then regard to:

(I) the uncertainty in, and smallness of, the amount of free hydrogen in the atmosphere near the ground;
(II) the opportunities for this hydrogen to unite with oxygen before diffusing to the stratosphere, and the further opportunities in the stratosphere;
(III) the absence of any direct evidence of the existence of a hydrogen atmosphere at great heights,

it seems reasonable to take into consideration as a serious possibility the assumption that the stratosphere contains no free hydrogen.«

It will appear from the following that the manner of propagation of radio waves clearly indicates that only very small, if any, quantities of hydrogen are present in the upper atmosphere. In the following we shall enter somewhat further into this question.

Finally it may be mentioned that *A. Wegener* has advanced the hypothesis that besides hydrogen there is present in the upper atmosphere a still lighter constituent, viz. a monatomic gas of molecular weight 0.4 which he calls geocoronium. We shall not enter further into *Wegener*'s hypothesis, but merely call attention to the statement by *Chapman* and *Milne* (l. c., p. 364) namely that *Jeans*[6] has shown the impossibility of the earth's atmosphere retaining a gas having a molecular weight as low as 0.4. The following investigations also show that no appreciable quantities of such a gas are present.

We shall in the following consider a series of different possibilities as to the composition and pressure of the atmosphere up to a height of about 200 km, examining in each individual case how far the experience concerning the propagation of radio waves may be found to agree with the conditions in an at-

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[1] *Chapman* and *Milne:* l c., p. 361.

[2] *Lord Rayleigh:* Phil. Mag. (6). Vol. 3, p. 416—422, 1902.

[3] *G. Claude:* C. R. 148, p. 1454, 1909.

[4] *J. H. Jeans:* The Dynamical Theory of Gases, p. 355 (2nd Ed. 1916). When *H. Lassen* (Jahrb. d. drahtl. Tel. Bd. 28, p. 111, 1926) writes that *Jeans* assumes 0.000033 per cent by volume, then this is not correct, but due to a printer's error in *Marx:* Handb. d. Radiologie. Bd. VI, p. 558 (1924).

[5] *A. Krogh:* Vid. Selsk. Mat.-fys. Medd. I, No. 12, Copenhagen (1919).

[6] *Jeans:* l. c. Chapter XV.

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mosphere of the nature considered. By these investigations it will be made evident that when the investigation is extended over the *entire* range of wave lengths, the manner of propagation of radio waves will determine quite narrow limits for the composition and pressure of the atmosphere at heigths between 80 and 160 km.

The various »atmiospheres« examined differ only in the amounts of hydrogen and water vapour they contain, in their different values of heights of effective mixing and, finally, in the vertical temperature gradient. The percentage of N_2, O_2, Ar, CO_2, Ne and He is in all cases the same as given in Table 2. In order to facilitate the comparison we have in Table 3 collected the constants for a number of the »atmospheres« examined.

Fig. 3 shows the distribution of pressure corresponding to the atmospheres tabulated in Table 3 and to the compositions given by *A. Wegener* (l. c., p. 46) and *W. J. Humphreys* (l. c., p. 64), and finally the pressure distribution used by *G. J. Elias*[1] is also shown. Fig. 4 shows the corresponding distributions of masses including also the one given by *Lindemann* and *Dobson* which is to be considered somewhat more closely.

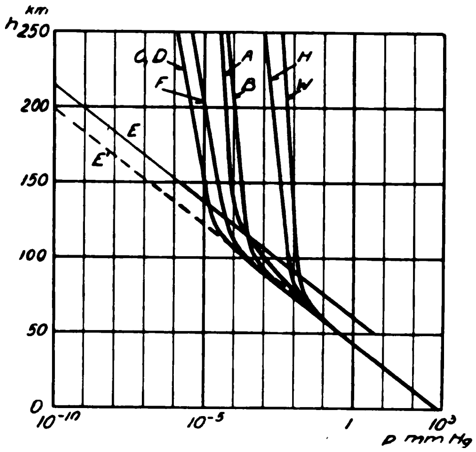

Fig. IV. 3. The Air Pressure as a Function of the Height, for various compositions of the atmosphere. The letters *A*, *B*, *C*, *D* and *F* refer to Table 3, and the curves *W*, *H* and *E* show the pressure distributions given by *Wegener*, *Humphreys* and *Elias*, *E'* being equal to 0.1 *E*.

From the following it will appear that in order to be able to explain in a satisfactory manner the propagation of radio waves we must assume the composition and pressure of the atmosphere between heights of 80 and and 160 km to be within or at any rate in the vicinity of the limits determined by *C*, *D* and *F*, and for the sake of briefness we shall call such an atmosphere a »radio atmosphere«. The question is now whether the composition and pressure conditions of the »radio atmosphere« may be considered reasonable, and in accord with experience in other fields.

Here it must be noted that up to heights of about 40 km, where the composition and pressure may be determined fairly accurately, the »radio atmosphere« coincides with the real one, but for heights above 80 km, our definite knowledge of the atmosphere is extremely small. Apart from the propagation of radio waves there are presumably only three other fields of experience which might possibly furnish fairly accurate information on this question, viz. (1) meteors, (2) the aurora borealis and (3) terrestrial magnetism. The dependency of terrestrial magnetism on the electric state of the atmosphere, however, has not yet been investigated far enough to allow us to draw any safe conclusions regarding the state of the atmosphere on this basis. We may therefore here limit ourselves to a reference to the remarks on this subject in the next chapter. Practically the same applies to the aurora

[1] *G. J. Elias:* Jahrb. d. drahtl. Tel. Bd. 27, p. 66—73, 1926.

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borealis. So little is known regarding the physical conditions that any conclusions concerning the composition and pressure of the atmosphere at great heights, based upon the various existing hypotheses, may be said to be at least uncertain. In this connection we shall just mention that *Vegard*'s interesting hypothesis[1] — namely that the atmosphere at heights above 100 km consists mainly of nitrogen in solid state (dust) — might perhaps after all prove serviceable even though a priori it might be thought incompatible with the requirements which we are led to regard as necessary for a satisfactory radio atmosphere. It would require, however, that *Vegard*'s hypothesis be modified at some points of minor importance[2].

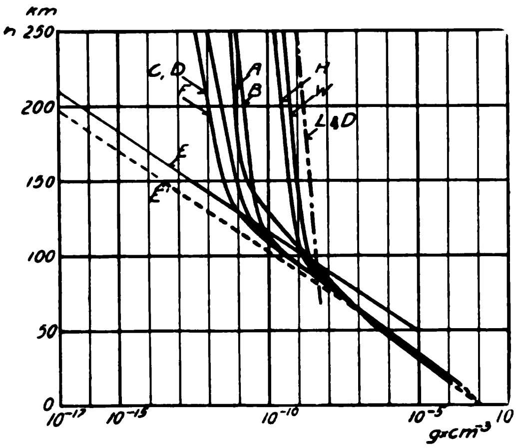

Fig. IV. 4. The Mass Density corresponding to the Pressure Curves shown in Fig. 2. The curve *L* & *D* shows the mass density adopted by *Lindemann* and *Dobson*.

The only question left relates to a knowledge of the mass density, temperature etc. of the upper atmosphere as obtained by a study of the meteors. *Lindemann* and *Dobson*[3] have treated the theory of meteors in a very interesting manner, and have thrown light on many of the characteristic features attached to this problem. Nevertheless, we do not think the density found by these authors to be of any decisive importance, and for the following reasons: *Lindemann* and *Dobson* take it for granted that by direct collision between a meteor with a velocity of about $u = 4 \cdot 10^6$ cm sec^{-1} and a slow moving air molecule of mass m, the meteor does not receive any appreciable portion of the kinetic energy $\frac{1}{2} mu^2$ lost by the collision, but that this energy is mainly used for ionization of the colliding molecule, for radiation, and for the increase in

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1 *L. Vegard:* Marx: Handb. d. Radiologie. Bd. VII, p. 578 (Fig. 16), 1924.

2 The nitrogen dust will exert practically no direct influence on the propagation of radio waves. Supposing therefore *that* the helium pressure decreases in the normal manner upward in the dust atmosphere, *that* the ultra-violet solar radiation liberates electrons from the dust particles, and *that* the said particles are minute, each particle consisting only of relatively few nitrogen molecules, then the conditions, from an electrical point of view, could probably be made to agree with those of the radio atmosphere, without the assistance of altogether too hazardous hypotheses. In our opinion, however, it will be better to postpone a more thorough treatment of this question until the various aurora-borealis hypotheses have been made a little clearer. We shall therefore here merely refer to the literature concerning this subject and mentioned in § 6 of the next chapter, and to: *J. C. McLennan* and *G. M. Skrum:* Proc. Roy. Soc. (A) Vol. 108, p. 501—513, 1925; *J. C. McLennan, J. H. McLeod* and *W. C. McQuarrie:* Proc. Roy. Soc. (A) Vol. 114, p. 1—22, 1927. *G. Cario*: Z. f. Phys. Bd. 42, p. 15—21, 1927.

3 *F. A. Lindemann* and *G. M. B. Dobson:* (1) Proc. Roy. Soc. (A). Vol. 102, p. 411—437, 1922; Vol 103, p. 339—342, 1923; *F. A. Lindemann:* (2) »Nature«. Vol. 118, p. 195—198, 1926.

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kinetic energy of the reflected parts of the colliding air molecule. Under this assumption the temperature increase of the meteor will therefore be quite insignificant, as long as the air pressure is very low. According to these authors, an air cap in front of the meteor catching the air molecules in the path of the meteor will not be formed until the meteor has descended so far that the air pressure rises to a certain value determined by the conditions at hand. The temperature of this air cap will necessarily be high, and a portion of the heat therefrom will then be transferred to the meteor, raising the temperature of the latter to such an extent that it will commence to evaporate appreciably. Not until then will the meteor become luminous. By means of the theory it is then possible to estimate the pressure at the height where the meteor becomes visible.

We cannot agree with *Lindemann*'s and *Dobson*'s view at least on one essential point. It appears to us without a doubt that an essential portion of the kinetic energy converted by the collision between the meteor and an air molecule moving at low velocity will be transferred to the meteor in the form of heat. The velocity of the air molecules relative to that of the meteor is in fact so high that it corresponds to the velocity of an ordinary ion which has passed through a difference in potential of several hundred volts. The experiences with charged and uncharged positive rays indicate clearly that under such conditions an essential portion of the kinetic energy of the particles will be converted into heat in the receiving »electrode«[1]. The experiences with ordinary electric vacuum tubes point precisely in the same direction. But if we assume that the main part of the kinetic energy of the collisions concerned is converted into heat in the meteor, we arrive at quite different results in respect to mass density in the higher atmosphere. We have had no opportunity to make any thorough examination of these questions but can only roughly estimate the mass density then found at great heights to be about 20 to 50 times smaller than calculated by *Lindemann* and *Dobson*. (See note at the end of this chapter).

We therefore feel justified in assuming that in other fields there is no reliable information showing that the composition and pressure of the real atmosphere differs from that of the so-called »radio atmosphere«. We are further of the opinion that the propagation of radio waves supplies a far more reliable basis for the determination of the state of the upper atmosphere than any of the other fields of experience hitherto used for this determination. In this connection we shall point out that: (1) radio waves may be emitted at will from the surface of the earth; (2) they return thereto in an observable manner; (3) the length of the wave used may be varied gradually from a few metres and up to several kilometres; and (4) the influence of the sun on the propagation of radio waves is very pronounced and easy to observe. In contrast to this is our lack of power to influence — not to say produce — auroras boreales, meteors or changes in terrestrial magnetism. As regards the aurora borealis we do not even know with certainty whether it be due to corpuscular radiation, and even though this appears to be most probable, we still do not know with certainty which particle is effective.

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[1] *Wien* und *Harms:* Handb. d. Experimentalphysik. Bd. 14, (W. Wien: Kanalstrahlen), 473, 1927.

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2. *The Mean Free Path of Electrons and Ions at Various Heights.*

The mean free path of the molecules in atmospheric air at 0° C. and at 760 mm pressure is assumed to be $l = 6 \cdot 10^{-6}$ cm [1]. At one mm pressure the free path will consequently be $l = 760 \cdot 6 \cdot 10^{-6} = 4.56 \cdot 10^{-3} \simeq 4.6 \cdot 10^{-3}$ cm.

The mean free path of the positive and negative ion is assumed to be of the same value, and we may consequently write:

$$l_{ion} = 4.6 \cdot 10^{-3} \cdot \frac{1}{p} \text{ cm} \quad (\text{p in mm Hg}). \tag{6}$$

The free path of an electron is assumed to be

$$l_{el} = 4\sqrt{2} \cdot l_{ion} = 2.6 \cdot 10^{-2} \cdot \frac{1}{p} \text{ cm} \quad (\text{p in mm Hg}). \tag{7}$$

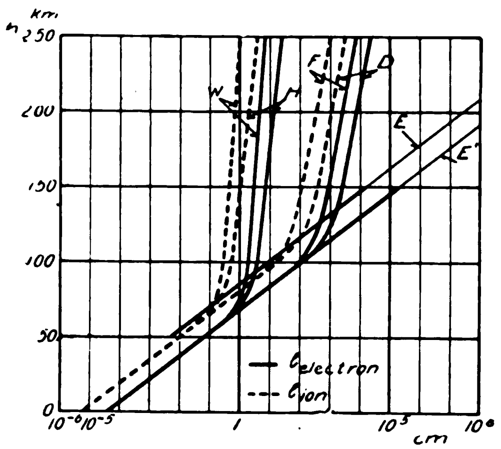

Fig. IV. 5. The Mean Free Path of Electrons and Ions at Various Altitudes. The Curves D, E, E′, F, H, W correspond to the same pressure distributions as mentioned in the text to Fig. 3 [2].

Since in our opinion the atmosphere at heights above 110 to 120 km consists mainly of helium we should for great heights really calculate the free path in the last mentioned gas. The ratio between the effective cross-sectional area of nitrogen and helium molecules in the case of collision with slow electrons may be assumed to be considerably greater than unity. But since a very accurate determination of the free path of the electrons or ions at great heights is not of great importance until later on, we have found it more advisable not to complicate the present calculations further by making the free paths depend not only on the pressure but also on the composition of the gas. It should be remembered, however, that at heights where the atmosphere consists mainly of helium we should have used a higher value of l_{el}. This question is taken up again in chapters IX and XI.

In the above mentioned paper by *G. J. Elias*[3] this author in agreement with *J. S. Townsend* and *H. F. Tizard*[4] writes

$$l_{el} = 3.2 \cdot 10^{-2} \cdot \frac{1}{p} \text{ cm.} \qquad (\text{p in mm Hg}) \tag{8}$$

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[1] *W. A. Roth* u. *K. Scheel:* Konstanten der Atomphysik p. 20 (1923).

[2] A numerical error appears to have slipped into *Elias*' paper. The free paths used by him presumably should have been in accordance with the line E′ in Fig. IV. 5. The discrepancy between the pressure distributions and free paths used by *Elias* and those used by us would then be correspondingly larger.

[3] l. c., p. 70.

[4] *J. S. Townsend* and *H. F. Tizard:* Proc. Roy. Soc. A. Vol. 88, p. 347, 1913.

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In a table *Elias* gives, furthermore, for various values of ω and for various heights, the value of the expression

$$r = \omega \frac{l_{el}}{U_0}, \qquad (9)$$

where U_0 is the mean thermic velocity of the electrons at 220° K. U_0 may therefore (see below) be determined as $9.15 \cdot 10^6$ cm sec^{-1}. From the given values of r the values of l_{el} used by *Elias* may consequently be computed.

On basis of l_{el} we may then by means of formula (8) compute the values of p used by *Elias*, and these pressure values are those mentioned on page 35 and shown in Figs. 3–5.

For the sake of comparison Fig. 5 shows, for some of the various pressure distributions mentioned above, the mean free paths of electrons as a function of the height h.

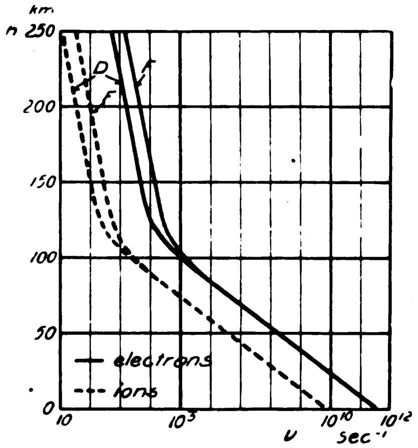

Fig. IV. 6. The curves $\nu_{el} = \left(\frac{U}{l}\right)_{el}$ and $\nu_{ion} = \left(\frac{U}{l}\right)_{ion}$ show the number of collisions per second suffered by an electron and an ion respectively.

The values of $\left(\frac{U}{l}\right)_{ion} = \nu_{ion}$ and $\left(\frac{U}{l}\right)_{el} = \nu_{el}$ corresponding to various atmospheres are shown in Fig. 6. ν_{el} and ν_{ion} are the number of collisions per second suffered by an electron and an ion respectively.

3. *The Mean Velocity of Electrons and Ions.*

The mean velocity of ions is taken as $U = 4.5 \cdot 10^4$ cm sec^{-1} at 0° C., corresponding to the thermic molecular velocity in a nitrogen atmosphere. The corresponding mass of the ions is about 50 000 times the mass of the electrons. These figures correspond very closely to those found for an N_2^+-ion. Other ions may of course also occur in the atmosphere — some being lighter ions originating from hydrogen, helium and other gases, some heavier ions from oxygen, ozone etc., while others again may be much heavier ions consisting of one of the former kinds in combination with one or more neutral molecules. It will hardly be necessary however, to pay much attention to these. In the first place there is presumably no hydrogen in the upper atmosphere; in the second place it is reasonable to assume that the heavier ions, which are of such great importance at high pressures and in the lower atmosphere, are relatively less numerous at the low pressures which will be especially dealt with here. At all events most of the following considerations would only be altered slightly, even though an essential part of the ions might have masses considerably more than 50 000 times that of the electron. This question will, however, be investigated further in sect. 12 of the next chapter.

The velocity of the electrons is more difficult to estimate. The thermic velocity for 0° C. is $U = 1.02 \cdot 10^7$ cm sec^{-1}. As mentioned above, *Elias* used the thermic velocity at 220° K and this velocity is $9.15 \cdot 10^6$ cm sec^{-1}. By passing through a voltage of V volts the electron acquires a velocity of $6 \cdot 10^7 \sqrt{V}$ cm sec^{-1}. The thermic velocity corresponds thus to a voltage of about $\frac{1}{36}$ volt.

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In order to estimate the approximate value of the velocity we shall first consider somewhat more closely the circumstances of the liberation of the electrons.

The velocity of the electrons, when set free, will depend on the frequency of the ionizing radiation, provided the radiation is an electro-magnetic wave-radiation, and in case of a corpuscular radiation it will depend on the velocity of the corpuscle, and will increase with increasing frequency and velocity. When the electron collides with a helium molecule, the collision will be elastic as long as the velocity is small, and the mean loss of energy of the electron for each collision[1] will then be extremely small. Collisions with nitrogen molecules will not be quite so elastic even in case of very slow electrons, but even so the loss is probably only very small. If on the other hand the velocity of the electron colliding with a nitrogen molecule is higher than the velocity corresponding to about 7 volts, then the electron may lose a considerable portion of its energy by the collision, as a rule not less than a portion corresponding to the said 7 volts. The mean velocity of the electrons must therefore at any rate be assumed to be lower than $6\cdot10^7\sqrt{7} = 1.59\cdot10^8$ cm sec^{-1}. Since the velocity of the electron after the last inelastic collision is below the above mentioned value and presumably has a mean value corresponding to about half the voltage, *i. e.* 3.5 volts, since a number of electrons presumably are liberated at relatively small velocity, and since there on an average is some loss of energy at each individual collision, even when the collision is elastic, then the mean velocity of the electrons must be assumed to be considerably lower than the value $1.59\cdot10^8$ cm sec^{-1}.

Considering then conditions of collisions with helium, that is with a loss of $\frac{1}{8500}$ of the kinetic energy per collision, the kinetic energy of the electron will be reduced to $\frac{1}{100}$ after about 18000 collisions. Since the time during which an electron is free is about 10 seconds at a height of 100 km, as it will appear from the following, and since the number of impacts per second at this level be about 10^5, it will be seen that the electron during the larger portion of its »free time« must have a velocity only slightly in excess of the thermic velocity. We therefore assume $U_{el} = 1.2\cdot10^7$ cm sec^{-1}, approximately corresponding to thermic velocity at 378° A = 105° C.

We are unable to agree with the assumption made by *W. G. Baker* and *C. W. Rice*[2], namely that $U_{el} = 4.85\cdot10^7$ cm sec^{-1} corresponding to a temperature of about 6000° A, and are of the opinion that these authors have somewhat overestimated the mean velocity of electrons.

In the above considerations the fact that the presence of electric fields may give the free electrons in the atmosphere a considerable velocity, has not been taken into account. Thus a field of one volt per cm would give the electrons a mean velocity in the direction of the field corresponding to about 0.025 volt at a height of 50 km, to about 25 volts at 100 km, and to about 250 volts at a height of 150 km. Here it should be noted, however, that if a field of such an intensity existed between the levels of 80 and 200 km, and if one electron was liberated per cm³ per second, then this field would cause such large dis-

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1 *G. Hertz* (Verh. Deutsch. Phys. Ges. 19, p. 268—288. 1917) has shown that a slow electron loses on an average $\frac{1}{8500}$ part of its kinetic energy by the collision with a helium molecule and about $\frac{1}{16}$ by collision with a hydrogen molecule. For nitrogen the loss may perhaps be estimated to be about $\frac{1}{1000}$.

2 *W. G. Baker* and *C. W. Rice:* Refraction of Short Radio Waves in the Upper Atmosphere. p. 18 in Complete paper (A. I. E. E., 1926).

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placements of charges inside a small fraction of a second that the field wolud become compensated. We may also mention that a field of about $3 \cdot 10^{-8}$ volts per cm is sufficient to neutralize the effect of gravity on a $\overline{N_2}$-ion. Electric fields are therefore not very likely to exert any considerable influence on the mean velocities of the electrons or ions within the region in which we are mainly interested.

The considerations here set forth do not preclude the existence, in the lower atmosphere, of quite powerful electric fields. The free paths are here so short that the velocities of ions and electrons, even in relatively intense fields, will be comparatively low.

The potential gradient in the lowest atmosphere and up to a height of about 10 km has been the object of many investigations. It appears from these that the magnitude of the gradient up to 1.5 km varies exceedingly, and that its normal value at the surface of the earth is about $+1$ volt cm^{-1}, the positive direction to be taken downward (the field corresponds to the charge of the earth being negative). Above 1.5 km the conditions are more regular[1]. The potential gradient at this height is about 0.25 volt cm^{-1} and decreases rather uniformly upward, so that at the maximum measured height, about 9 km, it is reduced to a value of about 0.04 volt cm^{-1} [2]. The measurements available thus tend to show that the electric field intensity approaches zero as the height increases.

4. *The Lifetime of Free Electrons.*

The »lifetime« of an electron we define as the mean length of time passed between the liberation of the electron and its recapture and conversion into a negative ion. We shall until further assume the degree of ionization to be so small that the probability of collision with a positive ion is negligible as compared with the probability of collision with neutral oxygen or water-molecules[3]. We further assume that the electrons under the existing conditions form only negative ions in combination with these molecules. The probability of a collision between an electron and an oxygen molecule resulting in the formation of a negative ion is denoted by s. Unfortunately the value of s has not heretofore been determined with any great accuracy. *V. A. Bailey*[4] assumes $s = 10^{-5}$, while *H. B. Wahlin*[5] indicates the value of s to be between $27.8 \cdot 10^{-5}$ and $11.5 \cdot 10^{-5}$ For reasons to be mentioned in the next chapter we have chosen *Bailey*'s value $s = 10^{-5}$.

We assume further the probability of formation of ions in case of collision between electrons and water molecules to be the same as between electrons and oxygen molecules, while the probability of ion formation by collision between a slow electron and a hydrogen, helium or nitrogen molecule may be considered equal to zero[6]. Thus it appears from *J. Franck*'s investigation[7] that

[1] *E. Mathias:* Traité d'électricité atmosphérique et tellurique, p. 71. (Paris 1924).

[2] *E. Everling* und *A. Wigand:* Ann. d. Phys. (IV) Bd. 66, p. 261—282. 1921.

[3] We assume consequently here that recombination may be disregarded. The cases where it must be taken into account will be treated later.

[4] *V. A. Bailey:* Phil. Mag. (6). Vol. 50, p. 825—843. 1925.

[5] *H. B. Wahlin:* Phys. Rev. (II). Vol. 19, p. 183. 1922.

[6] See for instance: *Geiger* und *Scheel:* Handb. d. Physik. Bd. XXII, p. 373—74. (*K. Przibram*). (Berlin 1926).

[7] *J. Franck:* Verh. d. Deutschen Phys. Gesell. 12. Jahrg., p. 613—620. 1910.

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the probability in case of nitrogen is negligible, and in case of hydrogen at any rate far less than for water vapour and oxygen. Subsequently *W. B. Haines*[1] has demonstrated that for quite pure hydrogen the probability is similarly negligible, and according to *J. Kasarnowsky*[2] the electron affinity of hydrogen is even negative. We cannot therefore support the viewpoint of *H. Lassen*[3], who in a quite recently published paper takes it for granted that the electrons are mainly caught by the hydrogen, while oxygen is of minor importance in this respect owing, in this author's opinion, to its very low pressure in comparison with hydrogen. Furthermore the hydrogen pressure is presumably still lower than assumed by *H. Lassen* as proved elsewhere in the present paper.

We introduce the symbol

$$\eta = \frac{p_{O_2} + p_{H_2O}}{p}, \tag{10}$$

thereby obtaining for the free time τ the expression

$$\tau = \frac{1}{s\eta\, v_{el}} = \frac{1}{s\eta} \cdot \left(\frac{l}{U}\right)_{el}. \tag{11}$$

Inserting here the value of l taken from formula (7) and η according to (10), we find:

$$\tau = \frac{2.6 \cdot 10^{-2}}{s\,(p_{O_2} + p_{H_2O}) \cdot U_{el}} = \frac{2.2 \cdot 10^{-9}}{s\,(p_{O_2} + p_{H_2O})}, \tag{12}$$

whence:

$$\frac{\tau_{h_1}}{\tau_{h_2}} = \frac{(p_{O_2} + p_{H_2O})_{h_2}}{(p_{O_2} + p_{H_2O})_{h_1}}, \tag{13}$$

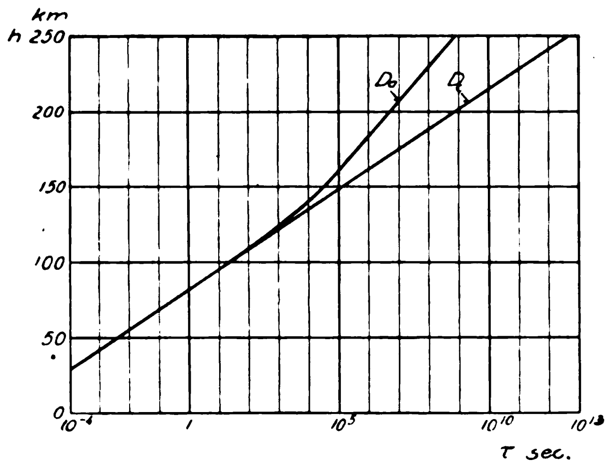

Fig. IV. 7. »Lifetime« of Electrons as a Function of the Height above the Earth's Surface. The notations refer to the atmospheres tabulated in Table 3. Curve D_c refers to the D atmosphere without any water vapour and D_a corresponds to $p_{H_2O} = 1.6 \cdot 10^{-3}$ mm at an Altitude of 20 km.

showing that the ratio between the values of τ at various heights is independent of both the magnitudes of the assumed values of s and U and of the total pressure.

In case the probability of ionization is different for oxygen and for water vapour, namely s_1 and s_2 respectively, then we have instead of (11) the following equation

$$\left.\begin{aligned} \tau &= \frac{p}{s_1 p_{O_2} + s_2 p_{H_2O}} \cdot \left(\frac{l}{U}\right)_{el} \\ &= \frac{2.6 \cdot 10^{-2}}{(s_1 p_{O_2} + s_2 p_{H_2O}) \cdot U_{el}} \end{aligned}\right\} \tag{14}$$

and, correspondingly,

1 *W. B. Haines:* Phil. Mag. (6). Vol. 30, p. 503—509. 1915.

2 *J. Kasarnowsky:* Zeitschr. f. Physik. Bd. 38, p. 12—21. 1926.

3 *H. Lassen:* Jahrb. d. drahtl. Telegraphie. Bd. 28, p. 109—113. Oktober 1926.

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$$\frac{\tau_{h_1}}{\tau_{h_2}} = \frac{(s_1\, p_{O_2} + s_2\, p_{H_2O})_{h_2}}{(s_1\, p_{O_2} + s_2\, p_{H_2O})_{h_1}}. \tag{15}$$

As far as we are aware, no further information is available regarding the value of s_2, and we have therefore assumed $s_2 = s_1 = s$.

The values of p_{O_2} and p_{H_2O} are calculated on the basis of the data contained in Table 3, and τ is then computed on the basis of formula (11). The values of τ for one of the atmospheres computed are then plotted in Fig. 7, where the notations refer to Table 3.

We wish to add that a more accurate knowledge of the value of s would be highly desirable, and that experiments attempting a direct determination of this factor are in preparation.

Note. Our attention has been drawn to a very interesting paper on the physical theory of meteors by *C. M. Sparrow*[1]. This author comes to the conclusion that it is not at all necessary to assume such high densities as those proposed by *Lindemann* and *Dobson* in order to explain the flashing of meteors. One sixth of the density assumed by *Humphreys* may be sufficient. We believe that even this density is greater than necessary in order to account for the flashing of meteors, but our reasons for this will be given elsewhere.

[1] *C. M. Sparrow:* Astrophys. J. Vol 63, p. 90-110. 1925.

CHAPTER V.

IONIZATION OF THE ATMOSPHERE[1].

1. *Recombination of Ions.*

If there are present, per cubic cm of air space, n_- negative and n_+ positive ions, then during the time element dt there will disappear by recombination $d(n_- + n_+)$ ions, where

$$d(n_- + n_+) = 2dn_- = 2dn_+ = 2\alpha n_- \cdot n_+ \cdot dt \tag{1}$$

and where α is the coefficient of recombination of the kinds of ions considered under the given conditions of air-pressure, temperature etc.

If the mean space charge is zero, then the number of positive and negative ions will be equal, so that

$$n_- = n_+ = n = \text{number of pairs of ions.}$$

In that case (1) may be written

$$\frac{dn}{dt} = -\alpha n^2. \tag{2}$$

(1) and (2) may also be considered as the equations defining the coefficient of recombination.

If, moreover, every second, in some way or other I ions are liberated per cubic cm, the resultant number of pairs of ions will be determined by

$$\frac{dn}{dt} = I - \alpha n^2. \tag{3}$$

The number of pairs of ions will consequently approach the stationary maximum value n_0 determined by:

[1] On the 28th of November, 1924, the questions dealt with in this chapter were made the object of a discussion in »The Physical Society of London« and »The Royal Meteorological Society« (»A Discussion on Ionization in the Atmosphere and its Influence on the Propagation of Wireless Signals«; Proc. Phys. Soc. Vol. 37, p. D1—D50, 1925; referred to in the following as »Discussion I«). The questions were also discussed in »The Royal Society« on the 4th of March, 1926 (Proc. Roy. Soc. (A), Vol. 111, p. 1—13, 1926; »Discussion on the Electrical State of the Upper Atmosphere«, referred to as »Discussion II«).

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$$n_0 = \sqrt{\frac{I}{\alpha}}. \tag{4}$$

If the outer ionization I ceases at the time $t = 0$, then according to equation (2) n is determined as follows:

$$\frac{1}{n} - \frac{1}{n_0} = \alpha t, \tag{5}$$

which may be given the form

$$n = \frac{n_0}{1 + \alpha n_0 t} = \frac{\sqrt{\frac{I}{\alpha}}}{1 + t\sqrt{\alpha I}}, \tag{6}$$

where n_0 is the value of n for $t = 0$, and where the last expression is used when n_0 is given by formula (4).

For large values of t such that $\alpha n_0 t = t\sqrt{\alpha I} \gg 1$, equation (6) may be written approximately:

$$n = \frac{1}{\alpha t}. \tag{7}$$

Thus after the lapse of sufficient time — this time being quite short, however, in many cases — n according to equation (7) becomes independent of the original number of ions n_0.

As to the magnitude of the coefficient of recombination, this quantity is for atmospheric air at 0° C. and 760 mm of the order of[1]

$$\alpha = 1.6 \cdot 10^{-6}. \tag{8}$$

The variation of α with temperature is not so large that it is necessary to take it into consideration here, but as far as the dependency of α on the air pressure p is concerned the case is different. There exist here two widely different points of view: *Mc Clung*[2] and others have found α to be independent of the pressure, *i. e.* $\alpha_p = \alpha_{760} = \alpha_0$, while *Langevin*[3] and others[4] believe to have proved that α is proportional to the pressure, *i. e.* $\alpha_p = \alpha_0 \frac{p}{760} = 2.1 \cdot 10^{-9} \cdot p$.

Since the magnitude of the coefficient of recombination at low pressures plays an important and in certain respects even a decisive part in the following considerations, we find it best to examine somewhat further the value of this quantity at very low pressures. According to the investigations made by *Langevin*, *Thirkill* and others α is most probably proportional to the pressure in the interval from somewhat below atmospheric pressure and down to pressures of about 10 mm. With the measurements available, however, we cannot with any degree of certainty estimate the value of α at the very low pressures around $1 \cdot 10^{-5}$ mm existing at a height of about 150 km. For estimating the value of α at such low pressures there exist, as far as we are aware, merely the above mentioned theoretical researches by *O. W. Richardson* and *Sir J. J. Thomson*. We shall here especially consider the work of the latter.

---

[1] *H. Geiger* und *K. Scheel:* Handb. d. Physik. Bd. XXII, p. 344 and 364 (Berlin 1926).

[2] *R. K. McClung:* Phil. Mag. (6), Vol. 3, p. 283—305. 1902.

[3] *P. Langevin:* Ann. de Chimie et de Physique (7), T. 28, p. 433—530. 1903.

[4] See for instance: *O. W. Richardson:* Phil Mag. (6), Vol. 10, p. 242—253. 1905.
H. Thirkill: Proc. Roy. Soc. (A). Vol. 88. p. 477—494. 1913.
Sir J. J. Thomson: Phil. Mag. (6). Vol. 47. p 337—378. 1924.

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According to *Thomson*, two ions $A(-e)$ and $B(+e)$ entering into each other's fields will recombine if after the collision they travel in closed paths about their common centre of gravity. The conditions for this may easily be deduced in the following manner: m_1, m_2 are the masses of A and B, respectively, u_1, u_2 their velocities and φ the angle between u_1 and u_2. The relative velocity u is then determined (see Fig. 1) by:

$$u^2 = u_1^2 + u_2^2 - 2u_1u_2\cos\varphi, \tag{9}$$

and the total kinetic energy K of the two ions relatively to the centre of gravity C is then:

$$K = \tfrac{1}{2} m_1 \left(\frac{m_2 u}{m_1+m_2}\right)^2 + \tfrac{1}{2} m_2 \left(\frac{m_1 u}{m_1+m_2}\right)^2 = \frac{1}{2}\frac{m_1 m_2}{m_1+m_2}(u_1^2 + u_2^2 - 2u_1u_2\cos\varphi). \tag{10}$$

If $K < \frac{e^2}{r}$, r being the distance between the ions, then A and B evidently cannot move infinitely far away from one another; their paths therefore become closed and they recombine. If, on the other hand, $K > \frac{e^2}{r}$, A and B will separate to an infinite distance from one another, and the ions will remain free.

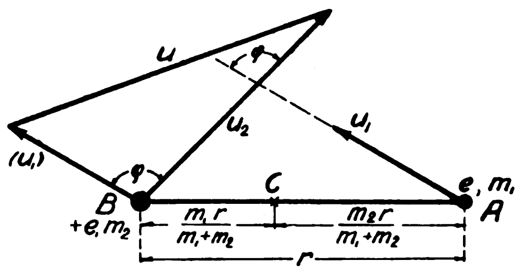

Fig. V. 1.
u is the Relative Velocity of the two Ions *A* and *B*.

We shall suppose that each ion has the mean kinetic energy corresponding to its absolute temperature T, namely

$$\tfrac{1}{2} m_1 u_1^2 = \tfrac{1}{2} m_2 u_2^2 = \beta T, \quad \text{where} \quad \beta = 2.02 \cdot 10^{-16}. \tag{11}$$

Substituting these values in (10) and assuming that $\cos\varphi$ has its mean value zero, we find that the ions will separate to an infinite distance when

$$K = \tfrac{1}{2}\frac{m_1 m_2}{m_1 + m_2}(u_1^2 + u_2^2) = \tfrac{1}{2} m_1 u_1^2 = \beta T > \frac{e^2}{r}. \tag{12}$$

Hence, if the ions at A and B possess a kinetic energy corresponding to the temperature, they will not recombine if their distance apart is greater than $\frac{e^2}{\beta T}$.

If we write

$$d = \frac{e^2}{\beta T} \tag{13}$$

then d will be the distance at which the average thermal energy will be just sufficiently large to prevent recombination.

When a positive and a negative ion are in the neighbourhood of one another, their kinetic energy due to the mutual attraction will on an average be larger than βT; but a collision between one of the ions within this part of the space and a neutral molecule will result in a reduction in the average kinetic energy of the ion. Supposing that this reduction decreases the value of the kinetic energy of the ions to βT, then recombination will occur, provided that collision

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takes place when the distance between the ions is smaller than d[1]. Assuming that each collision occurring for $r < d$ results in recombination, and assuming further that $p < 10$ mm, and accordingly $d \ll l$, where l is the free path of the ions, then it may be shown, without difficulty, (see *Thomson:* l. c., p. 338 to 340) that the coefficient of recombination α gets the value

$$\alpha = 2\pi d^3 \left(\frac{1}{l_1} + \frac{1}{l_2}\right) \sqrt{u_1^2 + u_2^2} = 4\sqrt{2}\pi \frac{u}{l} d^3, \tag{14}$$

where the last expression applies to the case where $l_1 = l_2 = l$ and $u_1 = u_2 = u$.

Since d and u are independent of the pressure while $\frac{1}{l}$ is proportional to p, then α will also be proportional to the air pressure.

In equation (14) l_1 and l_2 indicate the mean lengths of the paths traversed by the ions between two such collisions by which they lose so large a portion of their kinetic energy that recombination takes place for $r < d$. For the heretofore considered ions of molecular magnitude we assumed each individual collision to be effective in this respect. This consideration, however, is not valid in the case of a recombination between an electron and a positive ion. The mass of the electrons is so small relative to that of the ions that the electron will acquire almost the entire increase in kinetic energy caused by the attraction between the electron and the positive ion. In this case the positive ion possesses no surplus of kinetic energy, so that its collision with neutral molecules cannot contribute anything towards recombination. Equation (14) is then reduced to

$$\alpha = \pi d^3 \frac{1}{l'_{el}} \sqrt{u_{el}^2 + u_{ion}^2} \simeq \pi d^3 \frac{u_{el}}{l'_{el}} \tag{15}$$

Further, the probability of an electron losing its entire surplus of kinetic energy at one single collision will be exceedingly small. *Thomson* calculates that for H_2^+-ions in hydrogen an average of 16 collisions between the electron and neutral molecules will be required to effect for the electron the decrease of kinetic energy needed for recombination, so that the energy free path l'_{el} will be about 16 times the geometrical one. Taking this into consideration it follows from equation (15) that $\alpha_{(\text{electron}, +\text{ion})} = 0.46 \cdot \alpha_{(-\text{ion}, +\text{ion})}$. Since the values of recombination coefficients found in the two cases do not differ much, and since the assumptions upon which the comparison is based are rather uncertain, we shall in the following assume the two coefficients to be equal, and write:

$$\alpha_{\text{Thomson}} = 4\pi\sqrt{2} \frac{u_{ion}}{l_{ion}} d^3. \tag{16}$$

Numerical values of α. We have used above, in Chapter IV, the values

$$l_{ion} = 4.6 \cdot 10^{-3} \frac{1}{p} \text{ cm}, \quad u_{ion} = 4.5 \cdot 10^4 \text{cm sec}^{-1} \text{ and } m_{ion} = 50000\, m_{el} = 45 \cdot 10^{-24} g.$$

To these correspond according to equations (11), (13) and (16): $d = 5 \cdot 10^{-6}$ cm and $\alpha = 2.2 \cdot 10^{-8}$ p. According to *Langevin* we have

[1] If $r < d$ it will be sufficient that the kinetic energy is reduced to $K \leqq \frac{e^2}{r}$.

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$$\alpha_{\text{Langevin}} = 1.6 \cdot 10^{-6} \frac{p}{760} = 2.1 \cdot 10^{-9} p, \tag{17}$$

so that at low pressures we have approximately

$$\alpha_{\text{Thomson}} \cong 10 \cdot \alpha_{\text{Langevin}} = 2.1 \cdot 10^{-8} p. \tag{18}$$

In the following we shall assume that the coefficient of recombination at low pressures will at any rate not be smaller than α_{Langevin}.

Although being convinced that *Thomson*'s formulas, within very wide ranges, furnish a correct representation of the dependency of the coefficient of re-

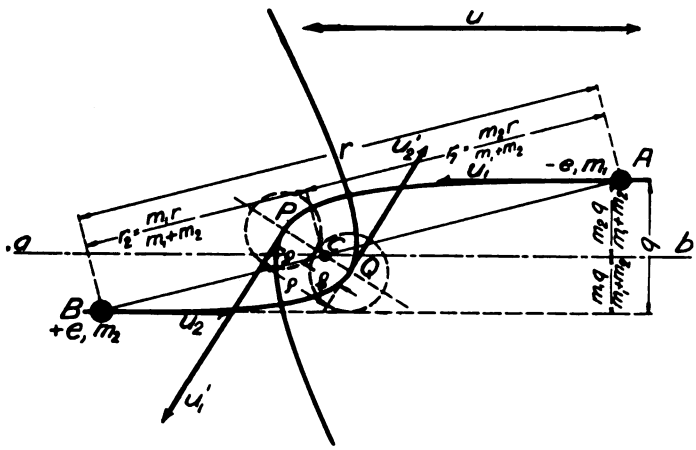

Fig. V. 2. The Figure shows the Paths Traversed by the two Ions *A* and *B* about the Common Centre of Gravity *C*.

combination upon the pressure, we still have some hesitation in applying these formulas all the way down to the very low pressures met with during the treatment of the present problem. We shall therefore consider the question of recombination at very low pressures from a somewhat different point of view.

Suppose two ions, $A(-e)$ and $B(+e)$ with masses m_1 and m_2 to have an initial distance r so great that the mutual attraction of the ions is inappreciable. The relative velocity u of the two ions is parallel to the line ab passing through their centre of gravity C, see Fig. 2. Supposing the latter to be stationary, the velocities of the two ions will be:

$$u_1 = \frac{m_2}{m_1 + m_2} u \quad \text{and} \quad u_2 = \frac{m_1}{m_1 + m_2} u. \tag{19}$$

At the points P and Q the ions have their minimum mutual distance $\varrho = \varrho_1 + \varrho_2 = CP + CQ$, and at the said points their velocities u'_1 and u'_2 are perpendicular to the line PQ. We have then:

$$\varrho_1 u'_1 = \frac{m_2}{m_1+m_2} q u_1, \quad \varrho_2 u'_2 = \frac{m_1}{m_1+m_2} q u_2, \text{ and } \varrho_1 = \frac{m_2}{m_1+m_2} \cdot \varrho, \quad \varrho_2 = \frac{m_1}{m_1+m_2} \cdot \varrho, \tag{20}$$

where q is the projection of the distance r of the two ions, on the perpendicular to their relative velocity u.

It is easily seen, that:

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$$\frac{u_1'}{u_1} = \frac{u_2'}{u_2} = \frac{q}{\varrho}. \tag{21}$$

The kinetic energy of ions in the position *PQ* is determined by:

$$\tfrac{1}{2} m_1 u_1'^2 + \tfrac{1}{2} m_2 u_2'^2 = \tfrac{1}{2} m_1 u_1^2 + \tfrac{1}{2} m_2 u_2^2 + \left(\frac{1}{\varrho} - \frac{1}{r}\right) e^2. \tag{22}$$

We now assume ϱ to decrease to a value in the neighbourhood of $3 \cdot 10^{-8}$ cm, *i. e.* to become of the same order of magnitude as the diameter of a molecule. In that case $\varrho \ll r$ and, simultaneously, $\frac{1}{2} m_1 u_1^2 \ll \frac{e^2}{\varrho}$ and $\frac{1}{2} m_2 u_2^2 \ll \frac{e^2}{\varrho}$. It then follows from equations (20), (21) and (22), that:

$$\frac{q}{\varrho} = \frac{e}{\sqrt{\beta T \varrho}}, \tag{23}$$

where the following substitution is made

$$\tfrac{1}{2} m_1 u_1^2 + \tfrac{1}{2} m_2 u_2^2 = \tfrac{1}{2} \frac{m_1 m_2}{m_1 + m_2} u^2 = \beta T. \tag{24}$$

Indicating by *C* the probability of collision between the two ions *A*(— e) and *B*(+ e), and by *c* the probability of collision when *A* and *B* are neutral, we have evidently

$$\frac{C}{c} = \frac{q^2}{\varrho^2} = \frac{e^2}{\beta T \varrho}. \tag{25}$$

The values previously found in Chapter IV for u_{el} and u_{ion} correspond to $T_{el} = 377^0$ *A*, and $T = 230^0$ *A*. Inserting further $\varrho_{el} = 1.6 \cdot 10^{-8}$ cm and $\varrho_{ion} = 3.2 \cdot 10^{-8}$ cm we find:

$$\left(\frac{C}{c}\right)_{el} = 187 \quad \text{and} \quad \left(\frac{C}{c}\right)_{ion} = 77. \tag{26}$$

We shall now proceed to determine the number of collisions between n_0 — ions and n_0 + ions, the aggregate number of pairs of ions being n_0. Assuming for the moment the positive ions to be converted into neutral molecules, the number of collisions between the n_0 negative ions and the n_0 neutralized positive ions will then be determined by:

$$A = \frac{u_0}{l} \cdot \frac{n_0^2}{N^0} = a \cdot n_0^2, \quad \text{where } a = \frac{u_0}{l N^0}, \tag{27}$$

u_0 being the velocity of the ions, *l* their free path and $N^0 = 4 \cdot 10^{16} \cdot p$ the total number of molecules per cubic cm.

Since u_0 is independent of *p*, while *l* is inversely and N^0 directly proportional to *p*, *A* and *a* will be independent of the pressure. Inserting the values of u_0 and *l* for electrons and ions, respectively, namely $u_{0el} = 1.2 \cdot 10^7$ cm sec^{-1}, $u_{0ion} = 4.5 \cdot 10^4$ cm sec^{-1}, $l_{el} = 2.6 \cdot 10^{-2} \frac{1}{p}$ cm and $l_{ion} = 4.6 \cdot 10^{-3} \frac{1}{p}$ cm, we find

$$a_{el} = 1.15 \cdot 10^{-8} \quad \text{and} \quad a_{ion} = 2.45 \cdot 10^{-10}. \tag{28}$$

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Multiplication of these figures by the values of the ratio $\frac{C}{c}$ as found above gives:

$$a_{el}^{o} = a_{el}\left(\frac{C}{c}\right)_{el} = 2.15 \cdot 10^{-6} \text{ and } a_{ion}^{o} = a_{ion}\left(\frac{C}{c}\right)_{ion} = 37.5 \cdot 10^{-9}, \tag{29}$$

and

$$A^0 = a^0 n_0^2 \tag{30}$$

will then be the number of collisions per second between the n_0 positive and the n_0 negative ions.

But only a small part of these collisions will result in recombination. Owing to the lack of reliable information concerning this, it would seem reasonable to assume the probability s_0 of a collision resulting in recombination to be equal to the probability s, found in Chapter IV, of a collision between an electron and an oxygen molecule resulting in the formation of an ion. At any rate there is hardly any reason for assuming s_0 to be greater than s. Some of the positive ions in combination with neutral nitrogen molecules will form still heavier ions which at the same time will be of a somewhat larger diameter than the one assumed above. According to the foregoing the probability of collision is proportional to the first power of the diameter of the ion. But as the probability of an electron forming an ion by collision with a nitrogen molecule is vanishingly small, the probability of recombination occurring by collision between an electron and such a heavy ion will presumably as a whole be smaller than the probability of recombination caused by collision with a positive oxygen ion. If we assume, as above, that s_0 is equal to s, the coefficient of recombination figured on the basis hereof will probably be slightly too large. But in order to be on the safe side we prefer in the following to estimate the value of the coefficient of recombination slightly too high rather than too low, and write therefore

$$\alpha_{el}^{o} = a_{el}^{o} \cdot s_0 = a_{el}^{o}\, s = a_{el}^{o} \cdot 10^{-5} = 2.15 \cdot 10^{-11}. \tag{31}$$

The probability of a collision between two approximately equally heavy positive and negative ions resulting in a recombination is without doubt considerably higher than s, but on the other hand the probability of collision in itself is according to equation (28) considerably lower. Since we are without reliable information regarding these facts, however, and since the value of α_{ion} is not of any decisive importance to the following considerations, we write simply

$$\alpha_{ion}^{o} = \alpha_{el}^{o} = \alpha^{o} = 2.15 \cdot 10^{-11}. \tag{32}$$

We assume in fact in the following $\alpha_{(electron, +ion)}$ to be equal to $\alpha_{(-ion, +ion)}$ throughout, since this will simplify the calculations considerably.

In *Thomson*'s deductions the above mentioned direct collisions are not taken into consideration. It will therefore be natural to increase the *Thomson* value of α by α^0, thus writing

$$\alpha_p = \alpha_0 \frac{p}{760} + \alpha^0 = 2.1 \cdot 10^{-8} p + 2.15 \cdot 10^{-11} \quad [p < 10 \text{ mm}]. \tag{32a}$$

For $p > 1$ mm, α^0 has practically no influence on the value of α_p, while for $p < 10^{-4}$ mm we have very closely $\alpha_p = \alpha^0$.

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In view of the foregoing considerations we must assume the estimated value of α^0 to be slightly too high. We therefore feel justified in assuming, that the coefficient of recombination at low pressures has a value somewhere between

$$\alpha_{\text{Langevin}} = \alpha_0 \frac{p}{760} = 2.1 \cdot 10^{-9} p, \tag{33}$$

and

$$\alpha_p = \alpha_{\text{Thomson}} + \alpha^0 = 2.1 \cdot 10^{-8} p + 2.15 \cdot 10^{-11}. \tag{34}$$

For the sake of comparison we have in Fig. 3, for the »atmosphere« marked *F* in Figs. IV 3 and 4 plotted the run of the value of the coefficient

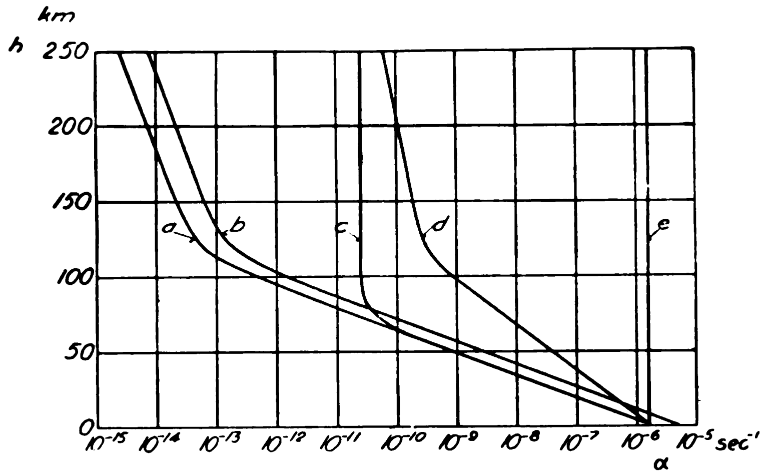

Fig. V. 3. A Comparison of the Values of the Coefficient of Recombination α as a Function of the Height *h* according to the Various Hypotheses Concerning the Relation between α and the Pressure. Curve (*a*) shows the value of α_{Langevin}, (*b*) α_{Thomson}, (*c*) $\alpha_{\text{Langevin}} + \alpha^0$, (*d*) $\alpha = \alpha_0 \sqrt{\frac{p}{760}}$, (*e*) $\alpha_{\text{McClung}} = \alpha_0$.

of recombination as a function of the height *h* above the surface of the earth, including also, besides the expressions (33) and (34), *McClung*'s value

$$\alpha_{\text{McClung}} = \alpha_0 = 1.6 \cdot 10^{-6}, \tag{35}$$

and the value

$$\alpha' = \alpha_0 \sqrt{\frac{p}{760}} = 1.6 \cdot 10^{-6} \sqrt{\frac{p}{760}}. \tag{36}$$

A more accurate knowledge of the value of α for various conditions would be of great importance to the further investigation of the problem here treated and, generally, to the question of the electrical state of the atmosphere. Experiments bearing on this subject are now in preparation and also the present investigation will clarify this question in various respects, for instance by indicating certain limits within which α, at low pressures, must be assumed to lie. We shall come back to these questions again later (see Chapt. IX).

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2. *Determination of the Number of Free Electrons.*

It will be shown in the following that the contribution of an electron towards the increase of the conductivity of the atmosphere and the reduction of the dielectric constant of the same may be very much greater than the corresponding contributions from a heavier ion. It is therefore necessary to keep separate accounts of the number of negative ions and of electrons.

We assume that τ is the mean free time of an electron, *i. e.* the mean duration of the period between the liberation of the electron and its conversion into a negative ion. We assume further that by the ionization I pairs of ions are liberated per cubic cm per second. The number of electrons per cc is denoted by n_e. By the conversion into negative ions there will then disappear $\frac{n_e}{\tau}$ electrons per second and per cubic cm. By recombination between electrons and positive ions $\alpha n n_e$ electrons will disappear per second and per cubic cm, the number n of ions being equal to the number of positive ions.

We may here restrict ourselves, however, to the treatment of two cases, namely partly the determination of the stationary state which occurs when the outer ionization has acted for a sufficiently long period and partly the determination of how the thus reached state is altered after the outer ionization has ceased.

We commence by considering the first mentioned problem. According to the assumption made the stationary number n_0 of pairs of ions is independent of the manner in which the total number of negative ions is made up of electrons and heavier ions. We have consequently

$$n_0 = \sqrt{\frac{I}{\alpha}}. \tag{4}$$

In accordance with what was said above, the number of electrons is determined by

$$\frac{dn_e}{dt} = -\alpha n_0 n_e - \frac{n_e}{\tau} + I. \tag{37}$$

The stationary number of electrons n_{e0} is consequently[1]

$$n_{e0} = \frac{I}{\alpha n_0 + \frac{1}{\tau}} = \frac{\alpha n_0^2}{\alpha n_0 + \frac{1}{\tau}} = \frac{n_0}{1 + \frac{1}{\alpha n_0 \tau}}. \tag{38a}$$

For $\alpha n_0 \tau \ll 1$ we have approximately

$$n_{e0} = \alpha n_0^2 \tau = I\tau \ (< n_0), \tag{38}$$

showing the number of electrons in this case to be independent of the value chosen for α.

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[1] In a quite recently published paper (Phys. Zeitschr. Bd. 27, p. 686. 1926) *H. Benndorf* assumes the state of ionization to be dependent only on I and α, and he leaves out of consideration the reduction in the number of electrons due to conversion of the latter into negative ions by recombination with oxygen or water molecules. It will be demonstrated later, however, that this reduction of the number of electrons is of great importance to the transmission problem.

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Later on we shall prove that at high altitudes I may in certain cases be assumed equal to

$$I = Ke^{-\frac{h}{H_{N_2}}}$$

where K is a constant independent of h, and H_{N_2} is the homogeneous height corresponding to nitrogen at the given temperature.

In that case the formula (38) may be written, — by means of the formulas (5) and (12) in Chapter IV

$$n_{e0} = K_0 \frac{e^{-\frac{h}{H_{N_2}}}}{(p_{O_2})\, e^{-\frac{h}{H_{O_2}}} + (p_{H_2O})\, e^{-\frac{h}{H_{H_2O}}}}, \tag{39}$$

where K_0 is a constant independent of h.

If we were to assume, in accordance with *H. Lassen*[1], that the electrons are bound only by hydrogen, we should instead of equation (39) get the following equation

$$n_{e0} = K_0' \cdot e^{-h\left(\frac{1}{H_{N_2}} - \frac{1}{H_{H_2}}\right)} \tag{40}$$

which latter with close approximation may be written

$$n_{e0} = K_0\, e^{-\frac{h}{H_{N_2}}} \tag{41}$$

owing to the fact that H_{H_2} is much greater than H_{N_2}.

We shall later see that the dependency of n_{e0} upon the altitude cannot be of this form.

The number n_{i0} of ions in the stationary state is

$$n_{i0} = 2n_0 - n_{e0} = n_0 \cdot \left(1 + \frac{1}{1 + \alpha n_0 \tau}\right). \tag{42}$$

For $\tau = 0$ the equation (38a) gives of course $n_{e0} = 0$, and for $\tau = \infty$ it gives $n_{e0} = n_0$.

We shall then determine the rate of decrease of n_e after ionization has ceased. In that case we have:

$$\frac{dn_e}{dt} = -\alpha n n_e - \frac{n_e}{\tau}, \tag{43}$$

where n is the number of pairs of ions, consequently also the number of positive ions, and according to equation (6) n is determined by

$$n = \frac{n_0}{1 + \alpha n_0 t}.$$

Equation (43) may therefore also be written

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[1] *H. Lassen:* Jahrb. d. drahtl. Telegraphie. Bd. 28, p. 109—113, 139—147. 1926.

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$$\frac{dn_e}{n_e} = -\left(\frac{\alpha n_0}{1+\alpha n_0 t} + \frac{1}{\tau}\right) dt, \tag{44}$$

and by integration of this last equation we find

$$n_e = \frac{\alpha n_0^2}{\alpha n_0 + \frac{1}{\tau}} \cdot \frac{1}{1+\alpha n_0 t} e^{-\frac{t}{\tau}} = \frac{n_0}{1+\frac{1}{\alpha n_0 \tau}} \cdot \frac{1}{1+\alpha n_0 t} e^{-\frac{t}{\tau}}, \tag{45}$$

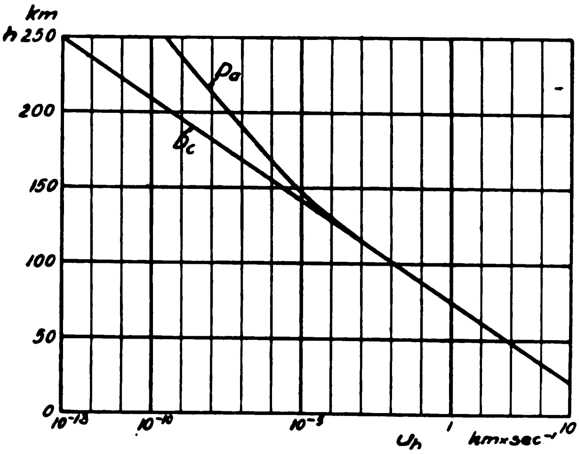

Fig. V. 4. The curves D_c and D_a show the upward Velocity u_h, after the cessation of the ionizing action, of the upper boundary of that part of the atmosphere, for which the Electron Density is Low. The curves correspond to the atmosphere »D«, curve D_c where no Water Vapour is present and D_a for $p_{H_2O} = 1.6 \cdot 10^{-3}$ mm. at an altitude of 20 km.

since for $t = 0$ we have

$$n_e = \frac{n_0}{1+\frac{1}{\alpha n_0 \tau}}.$$

For n_0 very large equation (45) gives approximately

$$n_e = \frac{e^{-\frac{t}{\tau}}}{\alpha t} < \frac{1}{\alpha t}. \tag{46}$$

Even in the case of infinitely high ionization the number of electrons at the time t after the ionizing action has ceased will therefore not exceed $\frac{1}{\alpha t}$, a value which is determined solely by the recombination, see equation (7).

If, on the other hand, $\alpha n_0 \tau \ll 1$ and $\alpha n_0 t \ll 1$, then equation (45) will be reduced to

$$n_e = I \tau e^{-\frac{t}{\tau}}. \tag{47}$$

In the equations (45) and (47) the last factor, the exponential function $e^{-\frac{t}{\tau}}$, will be the deciding one when t reaches values for which $t \geqq 10\,\tau$. It might be objected, partly that n_0 will assume various values at various altitudes, although within wide limits this variation is not considerable, partly that α will depend to a great extent on the height, and finally that t enters into the denominator of the intermediate factor of equation (45). But all these facts do not count much in comparison with the influence exerted by the exponential function when $t \geqq 10\,\tau$. The velocity u_h of the upward displacement of the low values of electron density after the cessation of the radiation will therefore mainly be determined solely by the fact that $e^{-\frac{t}{\tau}}$ continues to be of the same value. Assuming therefore τ to be a function of h and putting $\tau = \tau(h)$ and $t = 10\,\tau(h)$, we have

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$$dt = 10 \frac{d\tau}{dh} dh$$

and, consequently,

$$u_h = \frac{dh}{dt} = \frac{1}{10 \frac{d\tau}{dh}}. \tag{48}$$

Fig. 4 shows, for the indicated distributions of atmospheric pressure, the values of u_h corresponding to the values of τ in Fig. IV, 7. It will be noted that up to a height of about 100 km the values of u_h are quite large, so much so that the density of electrons almost instantaneously decreases to zero when the ionizing action ceases. For heights over 150 km, on the other hand, u_h will be very small, so that the boundary between the lower atmosphere containing few electrons and the higher atmosphere containing a greater number of electrons tends to remain stationary.

According to equations (6) and (45) the number n_i of ions at the time t after the cessation of the ionizing action is determined by

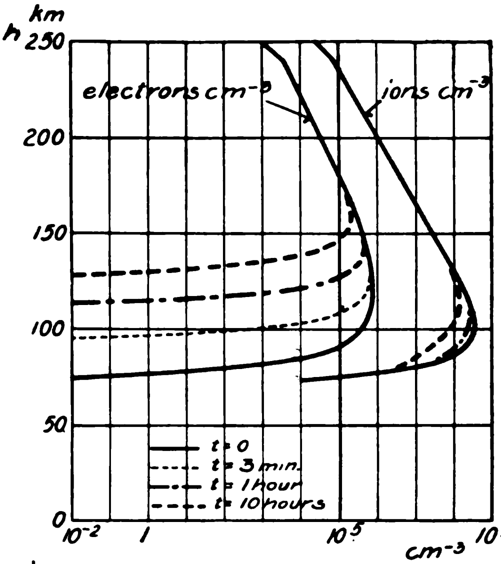

Fig. V. 5. The Number of Ions and Electrons during Stationary Ionization as well as 3 Minutes, 1 Hour and 10 Hours after the Cessation of the Ionization.

$$n_i = 2n - n_e = \frac{n_0}{1 + \alpha n_0 t}\left(2 - \frac{1}{1 + \frac{1}{\alpha n_0 \tau}} e^{-\frac{t}{\tau}}\right). \tag{49}$$

In order to illustrate this and also to give an impression of how rapidly the positive and negative ions disappear by recombination, see equations (6) and (7), we have shown in Fig. 5 the variation of the number of ions and electrons both for stationary ionization and also after 3 minutes, 1 hour and 10 hours have passed since the cessation of the same.

3. *The Absorption of External Radiation in the Atmosphere and the Ionization Produced thereby.*

A vertical radiation S penetrates down through the atmosphere which, preliminarily, is supposed to be of the same composition throughout the entire height. The original intensity of radiation is indicated by $S_\infty = S_{h \to \infty}$, where h is the altitude. So far we make no assumptions concerning the nature of the radiation, only that the attenuation dS suffered by the radiation while passing through an air layer of height dh is proportional to the intensity of the radiation, to the thickness of the layer and to the partial pressure p of the absorbing gas. We write consequently

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$$dS = AS\,e^{-\frac{h}{H}}\,dh \quad (50)$$

where Adh is that fraction of the radiation which would be absorbed in the layer dh if the latter was situated at the surface of the earth. H is as usual the height of the »homogenous atmosphere«, see Chapter IV, Table 2.

The equation (50) is satisfied by[1]

$$S = S_\infty\, e^{-AHe^{-\frac{h}{H}}} = S_0 e^{AH\left(1-e^{-\frac{h}{H}}\right)} \quad (51)$$

where S_0 is the Intensity of radiation at the surface of the earth.

The absorption at the height h is determined by

$$\frac{dS}{dh} = AS\,e^{-\frac{h}{H}} = AS_\infty\, e^{-\left(\frac{h}{H}+AHe^{-\frac{h}{H}}\right)}. \quad (52)$$

We assume the number I_h of pairs of ions liberated at the altitude h per cubic cm and per second to be proportional to $\frac{dS}{dh}$, *i. e.* proportional to the loss of energy of the radiation:

$$I_h = k\frac{dS}{dh} = kAS\,e^{-\frac{h}{H}} = kAS_\infty\, e^{-\left(\frac{h}{H}+AHe^{-\frac{h}{H}}\right)} = \varkappa A\, e^{-\left(\frac{h}{H}+AHe^{-\frac{h}{H}}\right)} \quad (53)$$

where $\varkappa = kS_\infty$, and $10^5\varkappa$ is the total number of electrons liberated per sec. in the atmosphere within a sunbeam column with one sq. cm cross-section.

I_h is maximum for

$$e^{\frac{h}{H}} = AH, \quad (54)$$

and

$$I_{max} = \frac{kS_\infty}{eH} = \frac{\varkappa}{eH} = \frac{\varkappa}{2.718\,H}, \quad (55)$$

or

$$\varkappa = eH\,I_{max} = 2.718\,HI_{max}. \quad (56)$$

The corresponding value of S is determined by:

$$S' = \frac{1}{e}S_\infty = \frac{1}{e}S_0 e^{AH} = 0.368\,S_0\, e^{AH}. \quad (57)$$

The value of h determined by equation (54) is the altitude corresponding to the maximum ionization and is indicated in the following by h_m.

The equation (53) may also be written in the form

$$I_h = eHAI_{max} e^{-\left(\frac{h}{H}+AHe^{-\frac{h}{H}}\right)} = I_{max}\, e^{\left(1+\frac{h_m-h}{H}-e^{\frac{h_m-h}{H}}\right)}. \quad (58)$$

For $h > h_m$ we have approximately

[1] *P. Lenard:* Sitzungsber. d. Heidelberger Akad. d. Wiss. 12. Abhandlung, Jahrgang 1911.

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$$I_h = eHAI_{max}\, e^{-\frac{h}{H}} = \varkappa A e^{-\frac{h}{H}}. \tag{59}$$

In case the radiation is not vertical but forms an angle φ with the perpendicular, and for distances within which the surface of the earth and the surfaces of uniform pressure of the air may be considered to be plane, we find exactly the same equations as above, replacing merely the coefficient of absorption A by the coefficient of absorption A' and $\varkappa$ by $\varkappa'$ determined by

$$A' = \frac{A}{\cos\varphi} \text{ and } \varkappa' = \varkappa \cos\varphi. \tag{60}$$

If we assume $\cos\varphi = 0.1$, thus making the coefficient of absorption ten times as great, then equation (54) shows the height of the maximum ionization to be increased by $2.3\,H$, while otherwise the shape of the ionization curve remains unchanged, as shown by equation (58).

If the value of I_h is known, the corresponding stationary number n_0 of pairs of ions, corresponding to the various assumptions concerning the dependency of the coefficient of recombination on the pressure, may be found by inserting into formula (4) the values of α given by equations (33) to (36) inclusive.

4. *Ionization of the Lower Atmosphere.*

The ionization of the atmosphere is best known in the vicinity of the surface of the earth, and direct measurements of the ionization exist only up to an altitude of about 15.5 km [1].

The number of ions per cubic cm is determined by measuring the conductivity of the air when exposed to the action of a constant electric field, and the number of pairs of ions per cubic cm has been found to be about five hundred at the surface of the earth. According to formula (4) this would correspond to an ionization of about 0.4 pairs of ions per cubic cm per second. Actually the ionization is considerably in excess of this [2], about 9 pairs of ions per c c per second, which would correspond to about 2400 pairs of ions per c c. One of the causes of this discrepancy must be the presence of a large number of very heavy ions besides the usual lighter ions. *Langevin* has measured the number of these heavy ions at Paris, and found it to be as much as 50 times the concentration of ordinary ions. The mobility of the heavy ions, however, is only about $^1/_{2000}$ of that of the ordinary ions, so that in spite of their great number they do not exert any appreciable influence on the conductivity and, therefore, do not count in the ordinary measurements of the number of ions.

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[1] *R. A. Millikan:* Ann. der Phys. (4), Bd. 79, p. 572—582. 1926.
R. A. Millikan and *J. S. Bowen:* Phys. Rev. (II). Vol. 27, p. 353—361. 1926.
R. A. Millikan and *R. M. Otis:* Phys. Rev. (II). Vol. 27, p. 645—658. 1926.

[2] See for instance:
B. Chauveau: Électricité atmosphérique, I (1922), III (1924). (Paris).
E. Mathias: Traité d'électricité atmosphérique et tellurique. (Paris 1924).
W. Kolhörster: Die durchdringende Strahlung in der Atmosphäre (Hamburg 1924).
» : Die Naturwissenschaften, p. 290—295, 313—319. 1926.
Victor F. Hess: Die elektrische Leitfähigkeit der Atmosphäre und ihre Ursachen. (Braunschweig 1926).
Geiger und *Scheel:* Handb. d. Physik. Bd. XIV 1927 (*G. Angenheister:* Atmosphärische Electricität. p. 405—441).

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These heavy ions, consisting of charged particles of dust or other inorganic or organic cores, are especially numerous where the air is very impure. They are at any rate so heavy that they exert no appreciable influence on the propagation of electrical waves.

The determination of the ionization, *i. e* the number of pairs of ions per c c liberated per second, may be effected by collecting, inside of a limited space, all the ions of a certain polarity on an insulated conductor charged with electricity of the opposite polarity. The rate of decrease of the voltage of the conductor will then be proportional to the number of ions liberated per unit of time.

The ionization of the lower atmosphere is largely due to the radio-active substances present in the upper strata of the earth and in the lower portion of the atmosphere. The ionization near the surface of the earth therefore decreases with increasing height above the earth but reaches a minimum at an altitude of one or two km, after which it again increases. This ionization increasing upwardly is due to other causes which will be further dealt with in the following.

The earth proper thus only affects the ionization of the atmosphere up to an altitude of a few kilometres, and since the normal ionization of the lower atmosphere is practically without importance to the propagation of the waves, as will be shown in the following, there will be no reason for entering into any further discussion of this ionization or of its dependency on the nature of the soil, the season, the meteorological conditions etc.

5. *Ionization Produced by a Highly Penetrating Cosmic Radiation.*

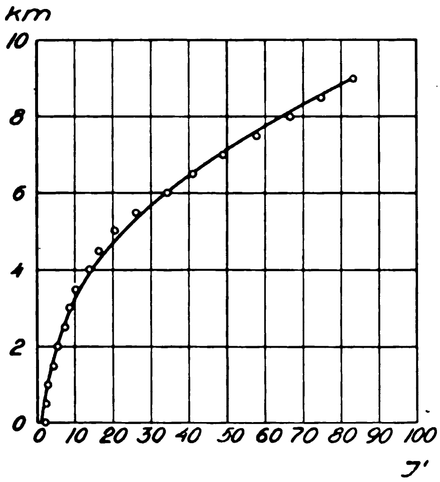

Fig.V.6. The Points Indicate the Results of *W. Kolhörster*'s Measurement of the Penetrating Radiation in Free Balloon. The ionization is determined for the interior of a closed metal vessel containing air of 760 mm pressure.

For the proof of the existence of this radiation and the investigation of the same we are indebted mainly to *Victor F. Hess*, *W. Kolhörster* and *R. A. Millikan*[1]. We shall here take as a basis *Kolhörster*'s measurements which cover a range of altitudes up to 9.3 km. and the results of which are represented in Fig. 6. The points plotted indicate the number of pairs of ions per c c liberated per second inside of a closed metal vessel containing air at a pressure of 760 mm. The curve shown in Fig. 6 corresponds to

$$\left. \begin{array}{c} I_1 = 460\, e^{-6 e^{-\frac{h}{7.2}}} \\ \text{(pairs of ions per c c per sec.)} \end{array} \right\} \quad (61)$$

The number I of pairs of ions liberated per c c per second in the real atmosphere may then be written

[1] See the above mentioned papers of the said authors as well as *W. Kolhörster:* Ann. d. Physik. Bd. 80, p. 621—628. 1926, and *V. F. Hess:* Phys. Zeitschr. 27, p. 159. 1926.

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$$\left. I = I_1 \frac{p}{760} = 460 \frac{p}{760} e^{-6e^{-\frac{h}{7.2}}} \quad \text{(pairs of ions per c c per sec.).} \right\} (62)$$

The stationary number of pairs of ions per c c is then given by equation (4). The values thus determined, together with the value of I, are shown in Fig. 7.

By means of formula (38a), which in this case is reduced to formula (38), we then determined the corresponding number of free electrons, the value of τ being taken from Fig. IV, 7. The values of n_0 and n_{e0} corresponding to the various values of α are also shown in Fig. 7.

This radiation is unceasing and there is no reason for considering the conditions after the radiation has ceased. In the present case we have only to deal with the stationary state.

That we have used *Kolhörster*'s measurements and not those of *Millikan*, although the latter cover a range of heights up to 15.5 km, while the former extend merely to 9.3 km, is due mainly to the following reason.

Both measurements relate to the ionization in a closed metal vessel, and therefore give only the ionization from the penetrating radiation. We may probably assume, however, that at these heights there exist other less penetrating radiations which are able, nevertheless, to ionize the air. Now *Millikan*'s measurements for heights between 5 and 15 km, give only about 25% of the ionization which might be expected according to *Kolhörster*'s experiments. And since we are unable at the present time to decide definitely which of the two series of measurements that gives the closer approximation to actual conditions we have for this reason chosen to start with the measurements giving the highest value for the ionization.

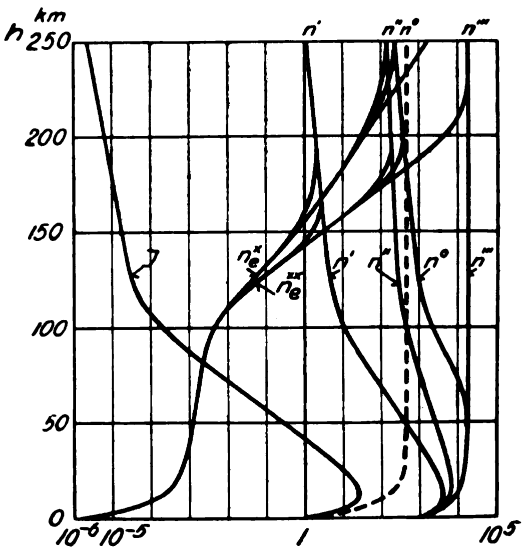

Fig. V. 7. Curve I shows the Ionization according to eq. (62), the broken line that according to eq. (61), the curves n', n'', n''', n^0 the number of pairs of Ions corresponding to the various values of α from eq. (35), (36), (33) and $\alpha = 2.1 \cdot 10^{-9}\, p + 4 \cdot 10^{-11}$ respectively; n_e^x the number of Electrons for $p_{H_2O} = 3.2 \cdot 10^{-3}$mm at 20 km height, and n_e^{xx} for $p_{H_2O} = 0$.

Since this very penetrating radiation is only of very minor importance to the present problem[1], we see no reason here for entering further into the results found by *Millikan* namely that only a relatively small fraction of the ionization at great heights is due directly to cosmic radiation, while the larger portion is produced by a secondary radiation released by this. Neither do we

[1] *H. Benndorf* expresses (Phys. Zeitschr. Bd. 27, p. 686—692, 1926) the opposite opinion, but as mentioned above we have not been able to accept the views of this author.

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find any reason for discussing here the circumstance that *Millikan* finds this cosmic radiation to possess a still higher power of penetration than that assumed by *Kolhörster*, nor for discussing the hypotheses of this or other authors concerning the origin of this very short-waved radiation. (*Millikan* assumes the wave length to be between about 0.00038 Å and a value about twice this. The shortest wave length corresponds to a frequency 10^7 times as high as that of the visible light)[1].

6. *Ionization Produced by External Corpuscular Radiation.*

The aurora polaris is generally assumed to be caused by such radiation[2], although agreement concerning the nature of the corpuscle has not been attained. Two possibilities in particular have been discussed, namely electrons or α-particles, but since this radiation according to our opinion does not belong to the factors of primary importance to the propagation of radio waves, we do not have to enter further into this question[3]. Since we, on the other hand, consider this corpuscular radiation to be the source of a portion of the disturbances to which the propagation of radio waves is exposed, we must here briefly mention a few facts concerning the aurora borealis phenomena. It has been demonstrated, especially by *C. Störmer*'s investigations, that the aurora borealis may penetrate down to an altitude of about 85 km, although most frequently its lower boundary is not less than about 100 km or more, and may be as much as 300 km or even 750 km, above the ground. This is

[1] *C. T. R. Wilson* (Proc. Camb. Phil. Soc. 22, p. 534—538. 1925. »Discussion II«, p. 5—6) has pointed out the very interesting possibility that the intense electric fields existing during thunder storms are able to impart to an electron such high velocities that by collision with a nucleus this electron would be able to release a very penetrating radiation of the above mentioned nature. Such radiation, however, would hardly play any considerable part in the ionization of the upper atmosphere.

[2] Out of the very extensive literature dealing with this subject we shall refer merely to the works of *C. Störmer* and *L. Vegard* mentioning especially:

C. Störmer: (1) Résultats des mesures photogrammétriques des Aurores Boréales observées dans la Norvège Méridionale de 1911 à 1922. (Oslo, 1926).
(2) Terrestrial Magnetism and Atmospheric Electricity, Vol. XXI, p. 45—56, 1916.
(3) Vol. XXII, p. 23—24, 97—112, 1917.
(4) Oslo Vid. Selsk. Skrifter mat. nat. Kl. No. 4, 1907; No. 3, 1913; No. 6 and 12, 1916.

L. Vegard and *O. Krogness:* (1) The Position in Space of the Aurora Polaris. (Oslo 1920).

L. Vegard: (2) Oslo Vid. Selsk. Skrifter mat. nat. Kl. No. 9, 1925; (3) Ann. d. Physik (IV). Bd. 79, p. 377—441, 1926. (4) »Das Nordlicht«, *Marx:* Handb. d. Radiologie, Bd. VI p. 505—596, 1924.

[3] Other authors, on the contrary, for instance *G. J. Elias* and *W. G. Baker* and *C. W. Rice* consider this corpuscular radiation to be of decisive importance to the propagation of radio waves, especially at night, see note on page 69. We have not been able to support this hypothesis, but find the criticism of *H. Benndorf* (Phys. Zeitschr. Bd. 27, p. 686, 1926), *H. Lassen* (Jahrb. d. drahtl. Tel. Bd. 28, p. 110, 1926), *F. A. Lindemann* (»Discussion II« p. 13) and others of the same to be fully justified. We shall revert to this question, however, in a later Chapter.

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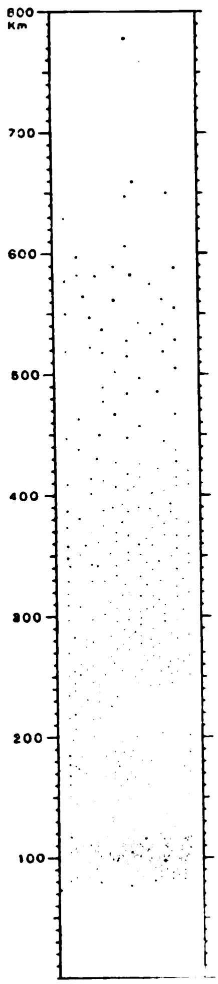
shown in Fig. 8 which is taken from one of *Störmer*'s papers[1]. The photographs of aurora borealis taken by *Störmer* and others show further the lower boundary of the same to be relatively uniform in many cases. *Störmer* has finally shown that the corpuscular radiation from the sun strikes the higher atmosphere not only in the neighbourhood of the magnetic poles and on the day-side of the earth, but may expand also over considerable portions of the night-side. These features play an essential part in the explanation of some of the characteristic irregularities to which the propagation of radio waves are subject, and we shall later come back to this subject.

The hypothesis[2] has also been brought forward that the sun's influence on the propagation of radio waves is due to electrically charged dust particles, flung out from the sun, owing to the high ray-pressure from this, and part of which strikes the outer atmosphere of the earth and causes the ionization of the same. Such



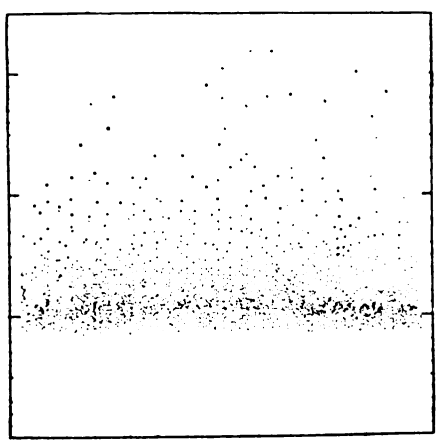


Fig. V. 8. Lower Limit of Aurora Borealis according to *Störmer.*

a hypothesis does not appear to us to be very probable. At any rate, *M.Tringali*[3] and *C. Nordmann*[4] have demonstrated that large eruptions on the sun may cause sudden variation in the magnetic state of the earth, and that these variations occur simultaneously with the eruptions becoming visible here on the earth, or at the most up to two minutes later, this being the estimated

[1] *C. Störmer:* (1) Fig. 10, p. 61.

[2] *J. A. Fleming:* Annales d. Postes, Télégraphes et Téléphones XII, p. 42—52, 1923.

[3] *M. Tringali:* Memoire del R. Osservatorio Astronomica al Collegio Romano, serio III, vol. VI, Parte I.

[4] *C. Nordmann:* Annales d. Postes, Télégraphes et Téléphones XII, p. 775—790, 1923.

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accuracy of the measurement. The agent effecting the transfer to the earth of these disturbances must therefore move either with the velocity of light or at least with four fifths of this velocity. The action must therefore presumably be due to the sunlight or possibly to an electron radiation. But it may be considered as out of the question that the action should be due to particles of dust, since the time necessary for these to travel to the earth would be at least 16 hours. As to a more detailed criticism of *Fleming*'s hypothesis we have to refer to *Nordmann*'s above mentioned paper. In this connection we shall further mention that *A. Schuster*[1] has proved that the daily variations in the magnetic field of the earth may be explained as being produced by the electric currents induced by the earth's magnetic field in a conducting atmosphere moving relatively to the *field*[2]. *A. Schuster* was thereby brought to assume a total conductivity of the ionizing layer amounting to $3 \cdot 10^{-6}$ (e. m. u.; cm). But we will have to postpone the discussion of this question until later on.

Very considerable contributions towards the enlightment of this question have recently been made from various sides[3].

Investigations made by *Tringali, Nordmann, Schuster, Chapman* and others indicate very strongly that the sunlight plays the most important part with respect to the ionization of the atmosphere at high altitudes, and our own investigations lead, as it will be shown, to the same result, but before proceeding to a further discussion of this ionization we shall very briefly touch on a possible ionization in the discontinuity surfaces of the lower atmosphere which under certain conditions may possibly exert some influence on the direction of propagation and the intensity of the waves.

7. *Ionization in Discontinuity Surfaces in the Lower Atmosphere.*

Two media M_1 and M_2 are in contact along the dividing surface *ab*, see Fig. 9. The conductivity is σ_1 and σ_2, for M_1 and M_2 respectively. The components of the electric field intensities perpendicular to the dividing surface are X_1 and X_2. The surface density ξ_{12} in the dividing surface will then be determined by

$$4\pi\, \xi_{12} = X_2 - X_1 \quad \text{(e. s. u).} \qquad (63)$$

For the stationary state, we must further have

$$X_1 \sigma_1 = X_2 \sigma_2, \qquad (64)$$

and equations (63) and (64) result in:

$$\xi_{12} = \frac{1}{4\pi} X_1 \left(\frac{\sigma_1}{\sigma_2} - 1\right) \cdot \quad \text{(e. s. u.)} \qquad (64a)$$

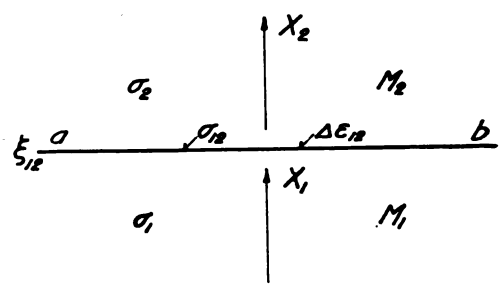

Fig. V. 9. Dividing Surface between two Homogeneous Media M_1 and M_2.

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[1] *A. Schuster:* The Diurnal Variation of Terrestrial Magnetism. Trans. Roy. Soc. (A). Vol. 208, p. 163—204, 1908.

[2] A hypothesis first proposed by *Balfour Stewart.*

[3] *S. Chapman:* »Discussion I«, p. D 38—45 and *Chree*, p. D 50. *Chapman*'s investigations show decisively that the ionization produced in the atmosphere by the sunlight influences the magnetic state of the earth.

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In the table below we have indicated the approximate values of ξ_{12} corresponding to $\frac{\sigma_1}{\sigma_2} = 0$, 10, 100 and $X_1 = 3 \cdot 10^{-3}$, 1 and 5 e. s. u.

ξ_{12} in e. s. u.

X_1 \ $\frac{\sigma_1}{\sigma_2} =$	0	10	100
$3 \cdot 10^{-3}$ e. s. u.	$-2.4 \cdot 10^{-4}$	$2.2 \cdot 10^{-3}$	$2.4 \cdot 10^{-2}$
1	-0.08	0.72	8
5	-0.4	3.6	40

The first value of the field-gradient, $X_1 = 3 \cdot 10^{-3} \cdot 300$ volts $\mathrm{cm}^{-1} = 90$ volts m^{-1}, corresponds to about the normal potential-gradient in fine weather. The second value $X_1 = 300$ volts $\mathrm{cm}^{-1} = 30000$ volts m^{-1} may for instance occur in the vicinity of thunder clouds, while the third value $X_1 = 150\,000$ volts m^{-1} may occur immediately below such clouds[1].

We shall take it for granted that all electricity in the dividing surface is present in the form of ions, since the life of the electrons at these high pressures is very short. We shall further assume all ions to be of uniform size, their mass being 50 000 times that of the electrons, see Chapter IV. Undoubtedly a relatively very considerable number of *Langevin* heavy ions will also be present, and our disregard of this fact really means that, other things being equal, we overestimate the conductivity σ_{12} of the dividing layer and also the decrease $\Delta\varepsilon_{12}$ of the dielectric constant. On the other hand ξ_{12} indicates the excess of one polarity of the surface charge, so that the number of ions per sq. cm of the surface will exceed $\frac{\xi_{12}}{e}$, where e as usual indicates the elementary charge of electricity. However, we assume the said number of ions to be equal to $\frac{\xi_{12}}{e}$, and the error thus committed tends to counteract the former one. The thickness of the dividing layer, however, is not vanishingly small but is, on the contrary, quite considerable, especially owing to diffusion of the ions. We shall come back to this later as well as to the determination of the conductivity and of the reduction in the dielectric constant due to this ionization in the dividing layers, and also to the reflection which may be caused by these layers.

Such discontinuity surfaces may occur, in clear air, as is well known, but they will be found especially at the transition between clouds and clear air. This agrees with the well known fact that the conductivity of the atmosphere

[1] *H. Norinder:* Undersökningar över det luftelektriska fältet vid åskväder. Tekniska meddelanden från Kungl. Vattenfallsstyrelsen. Serie E. No. 1 (Uppsala 1921).
C. T. R. Wilson: (1) Phil. Trans. Roy. Soc. (A), Vol. 221, p. 73—115. 1920. (2) Proc. Phys. Soc. Lond. Vol. 37, p. D 32—37. 1925.

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is generally lower in case of fog than when the air is clear, frequently even much lower[1].

8. *The Ionization caused by Sunlight.*

V. Schumann[2], *Th. Lyman*[3] and others have shown that the absorption in hydrogen is extremely small even down to the shortest waves of the ultra-violet spectrum with which it has been feasible to work. *Lyman* (l. c. p. 69) has further shown the same to be true in the case of helium. Oxygen, on the other hand, shows a decided absorption range when the wave length is shorter than $\lambda 1850 Å$ (*Lyman,* l. c. p. 63), as does nitrogen inside the *Schumann* range, although there is some uncertainty as to the nature and extent of this absorption[4]. Furthermore *Kreusler* (l. c. p. 420) has shown that water vapour is absorbing freely within the *Schumann* range.

Finally *P. Lenard*[5] has demonstrated that a mixture of pure nitrogen and pure oxygen is ionized when exposed to a powerful ultra-violet radiation of a wave length between $\lambda 1800$ and $\lambda 1200 Å$, while the ionization of the said gases is far less pronounced inside the range of still shorter waves ($\lambda 1200$ to $\lambda 900$ *Å U*).

It is therefore natural to assume[6] (1) that a portion of the ultra-violet radiation from the sun is absorbed by the nitrogen, oxygen and water vapour present in the higher atmosphere, and that this absorption is accompanied by an ionization; (2) that any hydrogen and helium which may be present does not contribute to this absorption or ionization.

Theoretically it would now be most correct to treat the three gases nitrogen, oxygen and water vapour separately, determining the absorption and ionization for each individual gas by means of the formulas deduced above in sect. 3. Such procedure, however, would seriously complicate the following considerations, and as our knowledge of the absorption and ionization caused by ultra-violet radiation after all is quite meagre, we shall in the following assume that the pressure in the mass of air in which the absorption takes place is reduced at the same rate, for increasing heights, as the pressure of the nitrogen. This assumption is contradictory to the fact that, other things being equal, the absorption will undoubtedly be greatest in oxygen. But as the pressure of the nitrogen at the heights here concerned is far higher than that of the oxygen, and as the water vapour also may give some ionization, we have thought this assumption to be the one giving the closest approximation to actual conditions. As an illustration of this we shall simply point out that the equivalent height for nitrogen (8.26 km) is intermediate between the equivalent heights for oxygen (7.23 km) and for water vapour (12.83 km).

In order to determine the ionization due to radiation from the sun on the basis of the formulas given in sect. 3 it will still be necessary to fix the value of

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[1] *Victor F. Hess:* Die elektrische Leitfähigkeit der Atmosphäre und ihre Ursachen, p. 30 (Braunschweig 1926).
E. Mathias: Traité d'électricité atmosphérique et tellurique, p. 145 (Paris 1924).

[2] *V. Schumann:* Ann. d. Phys. (IV), Bd. 4, p. 642—645, 1901.

[3] *Th. Lyman:* The Spectroscopy of the Extreme Ultra-Violet, p. 70 (London 1914).

[4] *Lyman:* l. c. p. 63; *H. Kreusler:* Ann. d. Phys. (IV), Bd. 6, p. 412—423, 1901; *V. Schumann:* Smithsonian Contributions. No. 1413.

[5] *P. Lenard:* Sitzungsber. d. Heidelberger Ak. d. Wiss. Math.-Nat. Kl. No. 28, 31, 32, 1910; No. 16, 24, 1911.

[6] Compare, however, sect. 9 of this chapter.

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the coefficient of absorption A and of $\varkappa = kS_{\infty} = eHI_{max}$, see equations (53) and (56). The former is decisive as to the depth of penetration into the atmosphere of the ionizing radiation from the sun, and may be determined on the basis of the existing measurements of the height of the lower boundary of the ionized layer of air. We shall later come back to these measurements. It will be found, by the way, that the manner of propagation of radio waves will give a quite reliable determination of the conductivity in the lower portion of the ionized layer of air, so that a rather sharp determination of the pressure at that height and, consequently, of the coefficient A may thereby be attained. The value of this coefficient furthermore depends on the hour, the season and the latitude of the locality concerned.

The value of the constant $\varkappa$, as we shall see in the following, is largely determined by the wave length of the shortest wave that returns regularly during the night.

When these two values have been fixed, and when the pressure and the composition of the air are given, it will be possible, on the basis of the following, to calculate the conductivity, the dielectric constant, the refractive index and the attenuation constant of the air for any frequency or wave length and for any altitude. On the basis thereof we may then determine the propagation of waves of any given length. If these consequences of the theory agree with experience, and if the assumptions regarding composition, pressure and ionization of the atmosphere can be considered plausible, then the theory must be said to give a satisfactory explanation of the manner of propagation of radio waves. Furthermore, since the above mentioned conditions will prove to be fulfilled — at any rate we regard it so —, and since the manner of propagation of radio waves, as dependent on the wave length, on the hour of the day or night and on the season, presents an extensive series of pronounced characteristics, the very fact that the theory gives a correct representation of all these complicated features will involve a very strong indication that the theory is substantially correct[1].

We therefore believe to have proved, in the following, that the ionization of the atmosphere caused by radiation from the sun is really the primary cause of the peculiar features of the propagation of radio waves and shall briefly consider a few of the objections having been raised against this point of view.

G. J. Elias[2], *H. Benndorf*[3], *J. A. Fleming*[4], *F. A. Lindemann*[5] and other in-

[1] *Sir Henry Jackson* (»Discussion II«, p. 7) expresses his impression of the great difficulties encountered by such theory in the following words: »Is a layer that will fulfill all these functions a physical possibility? If not, what is the cause of these variations?«

[2] *G. J. Elias* writes: »Diese Strahlen [α-Strahlen], können also ebensogut die Nacht wie die Tagesseite der Erde erreichen. Wir müssen deshalb annehmen, dass dadurch eine permanente Ionisation der Atmosphäre verursacht wird. Zweitens ist die kurzwellige Sonnenstrahlung im gleichen Sinne wirksam, aber natürlich nur bei Tag. Man müsste also bei Nacht eine, bei Tag aber zwei ionisierte Schichten annehmen«. (Jahrb. d. drahtl. Telegraphie, Bd. 27, p. 66, 1926).

[3] *H. Benndorf:* Phys. Zeitschr., Bd. 27, p. 686—692, 1926.

[4] *J. A. Fleming:* Ann. des Postes, Télégraphes et Téléphones. XII, p. 42, 1923. (»L'ionisation requise ne peut pas non plus être produite par la lumière solaire, puisqu'en se recombinant la nuit, les ions la feraient cesser«.)

[5] »Discussion II«, p. 13: »The main problem, namely to explain the production of the

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vestigators express the view that ionization due to the ultra-violet in the sunlight is not in itself sufficient to furnish an explanation of the propagation of radio waves, but this standpoint is, in our opinion, based on a mistake, namely that these authors assume the ionization caused by sunlight during daytime to disappear practically completely at night. As shown above in sects. 2 and 3, and as further explained in the following, this is not the case.

Much the same may be said of the idea that the ionized portion of the atmosphere should normally be divided into several independent layers, an opinion which appears to be very widespread. Reference shall here be made only to »Discussions I and II« where this train of thought has frequently been set forth[1]. We cannot see any reason for assuming such a separation to exist. The features in connection with the propagation of radio waves may be explained in a fully satisfactory manner, as shown below, on the basis of the ionization of one single coherent portion of the atmosphere. The reasons for the existence of two layers as given by *S. Chapman*[2] and based on the action of terrestrial magnetism can hardly be considered decisive.

Although solar radiation is the principal factor influencing the propagation of radio waves, this fact does not by any means prevent the ionization caused by corpuscular radiation from disturbing and modifying the ionization created by the sun. In the following we shall mention some few typical cases due, undoubtedly, to such interference.

In addition, very important variations are taking place in the ultraviolet radiation from the sun, as well as in all other solar radiation. This variation will of course also manifest itself in the ionization caused by the sun.

Another point ought to be mentioned. The ionizing radiation from the sun will not be mono-chromatic and its absorption coefficient A, therefore, will not be constant. However, in order to simplify the following calculations we neglect these differences in the value of A and take some constant mean value. This simplification will not cause any noticeable deviation from the true distribution of the ionization except in one respect: It will result in a too sudden falling off in the lowest part of the ionization. In order to at least partly compensate for this discrepancy we have in the following, in determining the distribution of the ionization in the lower region of the same made the thick-

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required ionization, therefore remains. In daytime it is simple enough for it can be attributed to solar ionization. This explanation breaks down at night, when most of the experiments were made. To attribute them to electrical effects such as auroræ seems fanciful, since these occur very much higher up and are much too erratic to account for such a regular phenomenon as the »Heaviside layer«. Ions caused by sunlight could scarcely persist in sufficient numbers at heights so low as are demanded.«

[1] *W. H. Eccles,* in closing »Discussion (I)« said: .... »The broad questions left unanswered are: How many conducting layers are there in the atmosphere? What is the position of each of these layers? Which of them function in the propagation of wireless waves« .... »Is the terrestrial magnetician's ionized layer the same as the wireless man's? We are still undecided. As a fact, the wireless man and the magnetician each want at least two conducting layers. Wireless phenomena can scarcely be understood at all without the aid of, firstly, a world-wide permanently ionized layer, and secondly a lower layer subjĕct to daily ionization; ....«, »Discussion I«, p. D. 48.

[2] »Discussion I«, p. D. 38—45, 46, 50.

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ness of the transition layer from very high to very low ionization about 10 km greater than the theoretical value for $A =$ constant.

The diffusion of the ions will also tend to smooth out any marked variations in the intensity of the ionization, but its influence will generally only be small.

9. *Ionization in the Upper Air and the Energy of Solar Radiation.*

The question about the energy balance of the ionization in the upper air has been discussed by *W. F. G. Swann*[1] and *S. Chapman*[2]. Both treat the radiation from the sun as black body radiation and assume the temperature of the sun to be 6000^0 C absolute. The whole energy E per cubic cm of the black body radiation is

$$E = \frac{8\pi}{c^3}\int_0^\infty \frac{h\nu^3}{e^{\frac{h\nu}{RT}} - 1}\cdot d\nu = \frac{48\pi R^4T^4}{c^3h^3}\cdot A, \tag{65}$$

where $A = 1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \cdots = 1.0823$. In formula (65) h is Planck's constant and equal to $6.55\cdot 10^{-27}$ erg. sec, ν is the frequency of the radiation, R is Boltzmann's constant and equal to $1.37\cdot 10^{-16}$ erg. deg^{-1}, and T the temperature.

The energy E_1 per c c contained between the frequency ν_1 and infinity is

$$E_1 = \frac{8\pi R^4T^4}{c^3h^3}\int_{\frac{h\nu_1}{RT}}^\infty \frac{x^3}{e^x - 1}\cdot dx \simeq \frac{8\pi R^4T^4}{c^3h^3}\int_{\frac{h\nu_1}{RT}}^\infty x^3e^{-x}dx. \tag{66}$$

Within the violet region $x = \frac{h\nu_1}{RT} > 1$ and $e^{-x} \ll 1$. The right hand side of (66) therefore gives a very close approximation to E_1. Hence

$$\frac{E_1}{E} = \frac{1}{6A}\cdot\int_{\frac{h\nu_1}{RT}}^\infty x^3e^{-x}\,dx = \frac{1}{6A}\cdot\left(\left(\frac{h\nu_1}{RT}\right)^3 + 3\left(\frac{h\nu_1}{RT}\right)^2 + 6\,\frac{h\nu_1}{RT} + 6\right)\cdot e^{-\frac{h\nu_1}{RT}}. \tag{67}$$

The solar constant S is equal to $1.35\cdot 10^6$ erg per sq. cm per sec.

Swann now assumes that only that part of the ultra-violet radiation for which $\lambda \leqq 1350$ Å, is available for ionization.

To $\lambda = 1350$ Å $= 1.35\cdot 10^{-5}$ cm corresponds $\nu_1 = 2.22\cdot 10^{15}$, $x = \frac{h\nu_1}{RT} = 17.7$, and $\frac{E_1}{E} = 2.1\cdot 10^{-5}$ while the ionization potential for this frequency is 9.2 volts, and the ionization energy $h\nu_1 = 1.456\cdot 10^{-11}$ erg.

The solar radiation S_1 available for ionization is accordingly $S_1 = \frac{E_1}{E}\cdot S =$ 28.35 erg per sq. cm per sec.

The rate G of production of ions as a result of the complete absorption of the radiation S_1 is $G = \frac{S_1}{h\nu_1} = 1.95\cdot 10^{12}$ pairs of ions per sec. within a sunbeam column of cross-section one sq. cm.

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[1] *W. F. G. Swann:* Terr. Magn. Atm. Elec. Vol. 21, p. 1—8. 1916.

[2] *S. Chapman:* Q. J. R. Meteor. Soc. Vol. 52, p. 225—236. 1926.

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According to *Schuster*'s theory of the daily variations in the magnetic field of the earth[1] the total conductivity of the atmosphere must be equal to $3 \cdot 10^{-6}$ (e. m. u., cm). *Swann* found that in order to account for this conductivity — assumed to be evenly distributed within a layer 300 km thick — the rate of production of ions must be $1.6 \cdot 10^{18}$ pairs of ions per sec. per sq. cm or about 10^6 times the value which could be obtained by the absorption of the ultra-violet radiation. To produce those $1.6 \cdot 10^{18}$ pairs of ions per sec. per sq. cm, would require an energy of $1.6 \cdot 10^{18} \cdot 1.46 \cdot 10^{-11} = 2.3 \cdot 10^7$ erg per sq. cm per sec., or about 17 times the total radiation from the sun!

Swann, however, in carrying out these calculations assumed the conductivity caused by one ion or one electron per cubic cm to be equal to about $4 \cdot 10^{-22}$ e. m. u., a value which at an altitude of 120 km is about 25 times smaller for the ion and for the electron about 25 000 times smaller than the values of the conductivities found by us. Besides, *Swann* assumed the recombination constant to be equal to $1 \cdot 10^{-6}$, while we have come to the conclusion that $\alpha \leqq 4 \cdot 10^{-11}$. By using our values of conductivities and of the recombination constant the necessary rate of production of ions would be about 10^{-13} times the value found by *Swann*, and with our values of the conductivities and of the recombination constant the energy of that part of the ultra-violet radiation for which $\lambda < 1350$ Å is amply sufficient to give the necessary total conductivity required by *Schuster*'s theory — at least as long as the influence of the magnetic field of the earth on the conductivity of the atmosphere is ignored, and neither *Swann* nor *Schuster* has taken this influence into consideration. We shall later return to this question.

Quite recently *S. Chapman*[2] has also discussed the energy relation of the ionization in the upper atmosphere. From the consideration of the »reflection« of radio waves he comes to the conclusion that the number N of electrons per c c at night must be at least 10^5 at an altitude of about 90 km. He assumes the conductivity caused by one electron per c c to be equal to $3 \cdot 10^{-19}$ e. m. u.[3] The specific conductivity, therefore, amounts to at least $3 \cdot 10^{-14}$ e. m. u. and *Schuster*'s value of the total conductivity will require a conducting layer of a thickness up to 1000 km, a value which is undoubtedly too great but nevertheless of the right order of magnitude, — but in this consideration no account has been taken of the influence on the conductivity of the earth's magnetic field, a point to which we shall come back later.

Chapman assumes the recombination constant to be equal to $1 \cdot 10^{-9}$ and comes to the conclusion that the rate of production of ions must be about $3 \cdot 10^{10}$ pairs of ions per sq. cm per sec. or about a hundred times less than the number G corresponding to the complete absorption of the sun's ultra-violet radiation at wave-lengths below $\lambda = 1350$ Å. *Chapman*, however, points out a new difficulty, namely, that according to *H. D. Smyth*[4] the lowest ionization potentials for nitrogen and oxygen are 16.9 and 15.5 volts respectively, corresponding to maximum wave lengths of radiation, for ionization, of 730 Å and 796 Å.

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[1] *A. Schuster:* Trans. Roy. Soc (A). Vol. 208, p. 163—204. 1908. See also Chap. IX.

[2] l. c.

[3] Our value of this conductivity is about $1 \cdot 10^{-18}$ e. m. u.

[4] *H. D. Smyth:* Proc. Roy. Soc. (A): Vol. 104, p. 121—134. 1923. Vol. 105, p. 116—128. 1924.

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Taking the previously mentioned values, 15.5 volts and 796 Å we get $x = \frac{h\nu_1}{RT} = 30$ and $\frac{E_1}{E} = 4.6 \cdot 10^{-10}$. Assuming black body radiation then the rate of production of ions cannot be greater than $G = 2.5 \cdot 10^7$ pairs of ions per sq. cm per sec. But in order to account for the necessary conductivity the value of G, according to *Chapman*, ought to be $3 \cdot 10^{10}$. *Chapman* overcomes this difficulty by assuming the ionization in the upper air to be connected with the formation of ozone[1]. This view may possibly be correct. We should like to point out, however, that according to *H. Sponer* and *R. T. Birge*[2] and *S. W. Leifson*[3] the oxygen molecule may be dissociated in neutral atoms at a wave length corresponding to 7.1 volts while the ionization potential of the neutral oxygen atom, according to *Franck* and *Jordan*[4] is about 13.6 volts. In agreement herewith *Hogness* and *Lunn* found the ionization potential of the O_2-molecule to be 13 ± 1 volts[5], somewhat lower than the value given by *Smyth*. Assuming the ionization potential to be 13.6 volts we get $\nu_1 = 3.3 \cdot 10^{15}$ and $x = \frac{h\nu_1}{RT} = 26.3$. With this value of x we have $\frac{E_1}{E} = 1.2 \cdot 10^{-8}$ and $G = 7.5 \cdot 10^8$ pairs of ions per sq. cm per sec. We have summarized the above results in the following table.

Table 4. Ionization in the upper air.

Solar constant $S = 1.35 \cdot 10^6$ erg per sq. cm per sec. Temperature of the sun $T = 6000^0$ C abs. and black body distribution of radiation.

V = the ionization potential in volts, and λ_1 the corresponding critical wave length.

G = the number of pairs of ions produced per sec. within a sunbeam column having a cross-sectional area of one sq. cm provided the total radiation energy between $\lambda = \lambda_1$ and $\lambda = 0$ is used for ionization.

V	9.2	13.6	15.5	Volts
λ_1	1350	910	796	Å U
G	$2 \cdot 10^{12}$	$7.5 \cdot 10^8$	$2.5 \cdot 10^7$	pairs of ions per sq. cm per sec.

The value of G should be:	according to *Swann*	about	$1.6 \cdot 10^{18}$	pairs of ions per sq. cm per sec.
	» » *Chapman*	»	$3 \cdot 10^{10}$	» » » » » » » »
	» » the writer[6]	»	$2 \cdot 10^9$ *	» » » » » » » »
	with maximum day ionization			
	at an altitude of 90 km	about	$7.9 \cdot 10^{10}$*	» » » » » » » »
	» » » » 80 »	»	$3.6 \cdot 10^{11}$*	» » » » » » » »
	» » » » 60 »	»	$7.3 \cdot 10^{12}$*	» » » » » » » »
	» » » » 50 »	»	$3.3 \cdot 10^{13}$*	» » » » » » » »

* Refer to the »atmospheres F and F'« and to a short-wave limit for night transmission of $\lambda = 18.9$ m, see Chapt. XI, sect. 1.

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[1] *Chapman:* l. c. p. 231; see also *G. M. B. Dobson* and *D. N. Harrison:* Proc. Roy. Soc. (A) Vol. 114, p. 521—541. 1927.

[2] See *J. Franck* und *P. Jordan:* Anregung von Quantensprüngen durch Stösse, p. 273 (Berlin 1926).

[3] *S. W. Leifson:* Astrophys. J. Vol. 63, p. 73—89. 1926. [4] l. c. p. 274.

[5] *Franck* und *Jordan:* l. c. p. 275. [6] The determination of this value will be discussed later.

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If we take 13.6 volts as the most probable value of the ionization potential the value of G is $7.5 \cdot 10^8$, and we are in the following led to the conclusion that the most probable value of the rate of production of ions is $G = 2 \cdot 10^9$, *i. e.* about 3 times the value which can be accounted for on the basis of black body distribution of the sun's radiation. We consider it probable that the radiation from the sun within the extreme ultra-violet region will be considerably in excess of that corresponding to a temperature of 6000° C and black body distribution. We believe it very unlikely that the high-speed particles — electrons, ions, α-particles, neutral atoms and molecules — which are expelled from the sun should not cause an ionization in the upper part of the chromosphere of such a character as to give a powerful radiation in the extreme ultra-violet region[1]. This view is further confirmed by the fact, that the conductivity of the air — according to *Chapman*[2] — is greater by about 50 per cent at sunspot maximum than at sunspot minimum. Besides, the value chosen for α is probably a little too great.

We therefore believe that all the evidence available confirms the view that the effective ultra-violet part of the energy radiated from the sun is just sufficient to account for the ionization which is required by the theory of radio wave propagation developed in the following.

10. *Ionization caused by Stellar Radiation.*

The influence on the propagation of radio waves of the ionization due to stellar radiation has not, at least as far as we are aware, hitherto been considered. It is, nevertheless, a factor of some importance in this connection.

The total (visible) light received by us from the stars is estimated to be equivalent to about 1000 first magnitude stars[3]. As the sun is 10^{11} times as bright as a first magnitude star, the sun is 10^8 times brighter than the total light from all the stars.

Some of the stars, however, are much hotter than the sun, and assuming black body radiation, they will therefore give relatively much more ultra-violet radiation. If we assume 5 per cent of the total radiation to come from stars at 18000°, 10 per cent at 12000°, 20 per cent at 9000°, 40 per cent at 6000°, and 25 per cent at 3000° [4], and take $\lambda_1 = 910$ Å as the critical wave length (ionization potential $V = 13.6$ volts) then the ratio $\frac{E_1}{E}$ between the ultra-violet and the total radiation will be about 10^5 times higher for the radiation from the stars than for the sun[5]. The total ionizing ultra-violet radiation from the stars should, according to this be $10^{-8} \cdot 10^5 = 10^{-3}$ times that from the sun. The ultra-violet radiation will, probably, be absorbed by the matter in interstellar space more

[1] See f. inst. *E. Pettit* and *S. B. Nicholson:* Astroph. J. Vol. 62, p. 202—228. 1925.

[2] *Chapman:* l. c. p. 229.

[3] *A. S. Eddington:* Bakerian Lecture. — Diffuse Matter in Interstellar Space. Proc. Roy. Soc. (A). Vol. 111, p. 424—456, 1926.
S. Chapman: »Monthly Notices«, Vol. 74, p. 450.

[4] *A. S. Eddington*: l. c. p. 430. Note.

[5] For comparison we quote the following figures from *Eddington*'s paper (Table III): Ionization potentials of 10.3 and 15.5 volts correspond to $\left(\frac{E_1}{E}\right)_{18000°} \Big/ \left(\frac{E_1}{E}\right)_{6000°} = 2.7 \cdot 10^3$ and $4.5 \cdot 10^7$ respectively.

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strongly than the visible light, but it is very difficult to estimate the value of this absorption with any degree of certainty, and as some of the stars probably have temperatures up to 25000° or 30000° C[1] we may have underestimated the ultra-violet radiation from the stars. Provisionally we, therefore, assume that the ultra-violet radiation from the stars is 10^{-3} times that from the sun. We believe, however, that the evidence to be derived from the experiences with radio wave propagation with regard to this question will be more trustworthy than any other available evidence in the matter.

The mean value of this ultra-violet »star radiation« is assumed to be constant, having the same value day and night, summer and winter but the individual values may possibly show considerable variations in analogy with corresponding radiations from the sun. The highly penetrating radiation mentioned in sect. 5 above may possibly be an extremely short-waved part of the ultra-violet radiation from the stars. Coming from sources having a higher temperature than the sun the penetrating power of this radiation is presumably greater than that of the sun's radiation[2].

The sun is $4.65 \cdot 10^5$ times as bright as the full moon[3] and the ultra-violet radiation from the moon is therefore considerably less than that from the stars, if our value of ultra-violet stellar radiation is not far too high.

Other radiations from the sky do not seem to play any important rôle[4].

11. *Ionization caused by Scattered Sunlight.*

Lord *Rayleigh* and others have shown the luminosity and blue colour of the sky to be due mainly to the scattering of the sunlight by the molecules of the air[5]. Lord *Rayleigh* proved that this scattering is proportional to λ^{-4} — λ being the wave length of the scattered light — and therefore especially effective in the case of violet and ultra-violet light.

The scattering does not play any important rôle with regard to the ionization in the sunlit part of the atmosphere. In the twilight regions, however, the scattered ultra-violet light may cause considerable ionization. With regard to the intensity of this scattering we may gather some information from the observed twilight illuminations at the earth's surface; such twilight illuminations being known for instance from the photometric measurements by *Kimball* and *Thiessen*[6] and given in Table 5 together with some other illumination intensities[7].

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[1] *A. S. Eddington*: l. c. p. 436.

[2] *A. S. Eddington*: l. c. p. 440, points out, however, that there will be a tendency for the ultra-violet radiation to be transferred to greater wave length, but he does not consider this shift to be of any great importance.

[3] *Kimball* and *Thiessen*: Monthly Weather Review. Vol. 44, p. 614, 1916, see also *W. J. Humphreys*: Physics of the Air, p. 550. 1920.

[4] *Lord Rayleigh*: Proc. Roy. Soc. (A). Vol. 99, p. 10—18, 1921; Vol. 100, p. 367—378, 1921; Vol. 101, p. 312—315, 1922; Vol. 103, p. 45—52, 1923; Vol. 106, p. 117—137, 1924; Vol. 109, p. 428—444, 1925.

[5] Lord *Rayleigh*: Phil. Mag. Vol. 47, p. 375—384, 1899. *R. J. Strutt* (Lord *Rayleigh*): Proc. Roy. Soc. (A). Vol. 94, p. 453—459, 1918. *Ch. Fabry*: J. de Physique. (V). Vol. 7, p. 89—102. 1917. *J. Cabannes*: Ann. de Physique. Vol. 15, p. 5—149, 1921.

[6] *Kimball* and *Thiessen*: Monthly Weather Review, Vol. 44, p. 614, 1916.

[7] From *W. J. Humphreys*: Physics of the Air, p. 550, (1920).

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Table 5. Relative Illumination Intensities on a fully exposed Horizontal Surface.

Source of illumination	Ratio to zenithal full Moon
Zenithal sun	465 000
Twilight at sunset or sunrise	1 598
Twilight; centre of sun 1^0 below horizon	1 453
» » » » 2^0 » »	727
» » » » 3^0 » »	358
» » » » 4^0 » »	150
» » » » 5^0 » »	53
» » » » 6^0 » »	19
(End of civil twilight)	
Twilight; centre of sun 7^0 below horizon	5.0
» » » » 8^0 » »	2.0
» » » » $8^0 40'$ » »	1.0
Zenithal full moon	1.0
Twilight; centre of sun 9^0 below horizon	0.75
» » » » 10^0 » »	0.40
Starlight	0.004

It is difficult, however, to estimate the amount of ultra-violet radiation scattered at higher altitudes, but in view of the small values of twilight illumination, when the sun is more than a few degrees below the horizon it is hardly probable that the corresponding ionization will have any great influence on the propagation of radio waves except for a short time after sunset and before sunrise, *i. e.* during the time in which the sun is just a few degrees below the horizon.

We shall have to come back to this question in connection with the discussion of summer night transmission.

12. *The Formation of Complex Ions of Medium Size.*

Up to the present we have only considered the influence of electrons and of ions consisting of one single molecule, and, no doubt, these two kinds of carriers are by far the most important in connection with the present problem. It is, however, necessary also to consider the question of formation of complex ions.

Attracting forces are acting between ions and uncharged molecules and these may therefore combine and thus form a complex ion. This complex ion may again combine with a neutral molecule to form a new and heavier complex ion and so on. With the scanty information at hand it is very difficult to form an estimate of the influence of this process on the conductivity of the air, at least at such low altitudes where the air contains a considerable amount of water vapor.

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We shall therefore simply assume that part of the conductivity due to the ionization which is caused by the very penetrating radiation as equal to 0.25 or 0.125 of the value it would have, if all the ions remained in the mono-molecular state. (At the surface of the earth the mean value of the conductivity is about $1.6 \cdot 10^{-25}$ e. m. u.[1]. If we take $I_{h=0} = 9$ pairs of ions per c c per sec. and $\alpha = 1.6 \cdot 10^{-6}$ we should, according to the results given in the next chapter, for mono-molecular ions get a conductivity about 12 times as high as the actual value. But the influence of the formation of complex ions on the conductivity will, no doubt, be less at higher altitudes. We have therefore, as mentioned above, only reduced the conductivity caused by the very penetrating radiation four or eight times.)

In the ionizations caused by solar and stellar radiations — and mainly found at higher altitudes than the above mentioned — we take the lifetime of a mono-molecular ion to be 400 times greater than the lifetime of an electron at the same altitude, that is, we put $\tau_{ion} = 400\, \tau_{elec}$. This is merely a very rough estimate, but τ_{ion} must at any rate be considerably greater than τ_{elec} since an ion makes fewer collisions per second and the probability of such a collision resulting in the formation of a complex ion is presumably less than the probability of a collision between an electron and an oxygen or a water molecule resulting in the formation of a mono-molecular ion.

Furthermore it will not have any serious consequences even if our estimate should be as much as 10 times too high or 100 times too low. We need not, therefore, investigate this point any further.

We assume the recombination constant to be the same for complex ions as for mono-molecular ions, and also the number of collisions to be the same for the two kinds of ions.

In order to calculate the number of the different kinds of ions, we introduce the following symbols:

n_e Number of electrons per c c.

n_{m+}, n_{m-} Number of respectively positive and negative mono-molecular ions pr. c c.

$n_m = n_{m+} + n_{m-}$.

n_{b+}, n_{b-} Number of respectively positive and negative bi-molecular ions per c c.

$n_b = n_{b+} + n_{b+}$.

$n = n_{m+} + n_{b+} = n_{m-} + n_{b-} + n_e$ Number of pairs of ions per c c.

$n_q = n_m + \frac{1}{a} n_b$ Equivalent number of mono-molecular ions per c c.

τ and τ_i Mean lifetime respectively of an electron and of an ion.

Since we assume α and v to be the same for bi-molecular and for mono-molecular ions, two bi-molecular ions will — as shown in the next chapter — cause the same changes in the dielectric constant and in the conductivity of the air as one mono-molecular ion. We therefore, generally take a equal to 2.

[1] *L. A. Bauer* and *W. F. G. Swann*: Researches of the Department of Terrestrial Magnetism. Carnegie Institution. Vol. 3, p. 406. (Washington 1917). *E. Mathias*: Traité d'électricité atmosphérique, p. 127. (Paris 1924).

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For the determination of the above mentioned quantities we have the following equations:

$$\left.\begin{aligned}
&\frac{dn}{dt} = -\alpha n^2 + I \quad \text{(a)} \qquad \frac{dn_e}{dt} = -\alpha n_e n - \frac{n_e}{\tau} + I \quad \text{(b)}\\
&\left.\begin{aligned}\frac{dn_{m+}}{dt} &= -\alpha n_{m+} n - \frac{n_{m+}}{\tau_i} + I \quad \text{(c)}\\ \frac{dn_{m-}}{dt} &= -\alpha n_{m-} n - \frac{n_{m-}}{\tau_i} + \frac{n_e}{\tau} \quad \text{(d)}\end{aligned}\right\} \frac{dn_m}{dt} = -\alpha n_m n - \frac{n_m}{\tau_i} + \frac{n_e}{\tau} + I \quad \text{(e)}\\
&\left.\begin{aligned}\frac{dn_{b+}}{dt} &= -\alpha n_{b+} n + \frac{n_{m+}}{\tau_i} \quad \text{(f)}\\ \frac{dn_{b-}}{dt} &= -\alpha n_{b-} n + \frac{n_{m-}}{\tau_i} \quad \text{(g)}\end{aligned}\right\} \frac{dn_b}{dt} = -\alpha n_b n + \frac{n_m}{\tau_i} \quad \text{(h)}\\
&\frac{dn_q}{dt} = -\alpha n_q n - \left(1 - \frac{1}{a}\right)\frac{n_m}{\tau_i} + \frac{n_e}{\tau} + I \quad \text{(i).}
\end{aligned}\right\} \quad (68)$$

The stationary values are, as usually, denoted by index 0 and we get:

$$\left.\begin{aligned}
&n_0 = \sqrt{\frac{I}{\alpha}} \ \text{(a).} \quad n_{e0} = \frac{\alpha n_0^2 \tau}{1 + \alpha n_0 \tau} \ \text{(b).} \quad n_{m+0} = \frac{\alpha n_0^2 \tau_i}{1 + \alpha n_0 \tau_i} \ \text{(c)}\\
&n_{m-0} = \frac{\alpha n_0^2 \tau_i}{(1 + \alpha n_0 \tau)(1 + \alpha n_0 \tau_i)} \ \text{(d).} \quad n_{m0} = \frac{\alpha n_0^2 \tau_i (2 + \alpha n_0 \tau)}{(1 + \alpha n_0 \tau)(1 + \alpha n_0 \tau_i)} \ \text{(e)}\\
&n_{b+0} = \frac{n_0}{1 + \alpha n_0 \tau_i} \ \text{(f).} \quad n_{b-0} = \frac{n_0}{(1 + \alpha n_0 \tau_i)(1 + \alpha n_0 \tau)} \ \text{(g)}\\
&n_{b0} = \frac{n_0 (2 + \alpha n_0 \tau)}{(1 + \alpha n_0 \tau_i)(1 + \alpha n_0 \tau)} \ \text{(h).} \quad n_{q0} = \frac{n_0 \left(\frac{1}{a} + \alpha n_0 \tau_i\right)(2 + \alpha n_0 \tau)}{(1 + \alpha n_0 \tau_i)(1 + \alpha n_0 \tau)} \ \text{(i)}
\end{aligned}\right\} \quad (69)$$

The values of the number of ions t seconds after the ionization I has ceased are denoted by index t and are determined by:

$$n_t = \frac{n_0}{1 + \alpha n_0 t} \quad \text{(a)} \qquad n_{et} = \frac{n_{e0}}{1 + \alpha n_0 t} \cdot e^{-\frac{t}{\tau}} = \frac{n_0}{1 + \alpha n_0 t} \cdot \frac{e^{-\frac{t}{\tau}}}{1 + \frac{1}{\alpha n_0 \tau}} \quad \text{(b)}$$

$$n_{mt} = \frac{n_{m0} \cdot e^{-\frac{t}{\tau_i}}}{1 + \alpha n_0 t} \left\{ 1 + \frac{1 + \frac{1}{\alpha n_0 \tau_i}}{1 + \frac{2}{\alpha n_0 \tau}} \cdot \frac{1 - e^{-\left(\frac{1}{\tau} - \frac{1}{\tau_i}\right)t}}{1 - \frac{\tau}{\tau_i}} \right\} \quad \text{(c)}$$

$$n_{bt} = 2n_t - n_{mt} - n_{et} \quad \text{(d)}$$

$$n_{qt} = \frac{n_{q0}}{1 + \alpha n_0 t} \left\{ 1 - \frac{1}{1 + \frac{1}{a \alpha n_0 \tau_i}} \left[(1+k)\left(1 - \frac{1}{a}\right)\left(1 - e^{-\frac{t}{\tau_i}}\right) - k\left(1 - \frac{\tau}{\tau_i}\left(\frac{1}{a}\right)\right)\left(1 - e^{-\frac{t}{\tau}}\right) \right] \right\} \quad \text{(e),}$$

where $k = \dfrac{1 + \frac{1}{\alpha n_0 \tau_i}}{1 + \frac{2}{\alpha n_0 \tau}} \cdot \dfrac{1}{1 - \frac{\tau}{\tau_i}} = \dfrac{1 + \alpha n_0 \tau_i}{2 + \alpha n_0 \tau} \cdot \dfrac{1}{\frac{\tau_i}{\tau} - 1}$.

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These formulas are used for the determination of the equilibrium of ionization in the atmosphere between the altitudes 70 and 110 km. Below 70 km the ionization does not play any important rôle in the propagation of radio waves and the conditions are too complicated to allow any exact treatment[1]. Above 110 km the influence of the electrons predominates, and the lifetime of the mono-molecular ions is so long that the bi-molecular ions are of minor importance. Above 110 km we, therefore, use the formulas given in section 2 above.

13. *The Ionization caused by Strong Electric Fields.*

Finally the meteorological conditions in the troposphere are likely to influence the ionization of the upper atmosphere, for instance through the electric fields which may be created at these high altitudes by electric charges in the troposphere. *C. T. R. Wilson*[2] has called attention to these circumstances by determining the mean value of the electric moment M of a thunder cloud, relatively to the earth, and has found $M = 3 \cdot 10^{16}$ e. s. u. cm. Above the thunder cloud, at an altitude of h km such a charge would give an electric field intensity E determined by

$$E = \frac{2M}{(10^5 h)^3} = \frac{60}{h^3} \text{ e. s. u} = \frac{18000}{h^3}. \qquad [\text{volts cm}^{-1}].$$

To $h = 60$ km corresponds $E = 0.08$ volt cm^{-1}, but since the free path of the electrons at this height is about 2 or 3 mm, an electric force of this magnitude will not be able to produce any appreciable ionization, and it would not be able to do so even if the assumed value of E were ten times as high. In addition to this, the electric field causes a displacement of the existing positive and negative ions which will tend to substantially compensate the field. This compensation would require only a surface density of about $2 \cdot 10^{-5}$ e. s. u. per sq. cm corresponding to about $5 \cdot 10^4$ ions per sq. cm, and since the density of ions at this height, during day and night, is about 10^4 ions per c c, the compensation of such an electric field would require an ion displacement of only about 5 cm. We cannot therefore attribute to the electric charges in the troposphere any considerable influence on the ionization of the upper atmosphere.

When we only in exceptional cases use the denomination »Heaviside layer« or »Kennelly layer«, our reason is that in our opinion the word »layer« is not an appropriate denomination for the highly ionized portion of the atmosphere extending without discontinuity from a height of about 80 km and up to a height of several hundred km.

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[1] See for instance: *E. v. Schweidler*: Wien-Berichte, math.-nat. Kl. IIa. Bd. 127, p. 953—967. 1918; Bd. 128. p. 947—955. 1919. *A. D. Power*: Journ. Frankl. Inst. Vol. 196, p. 327—352. 1923. *J. J. Nolan, R. K. Boylan,* and *G. P. de Sachy*: Proc. Roy. Irish Ac. (A) Vol. 37. p. 1—12. 1925).

[2] »Discussion I«, p. D. 32; »II«, p. 5.

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CHAPTER VI.

THE INFLUENCE OF ELECTRONS AND IONS ON THE CONDUCTIVITY AND DIELECTRIC CONSTANT OF THE ATMOSPHERE.

1. *The Influence of Electrons and Ions on the Conductivity of the Atmosphere at Various Heights.*

As an introduction to and a check on the following calculations we shall begin by determining the conductivity in the case of constant fields. We therefore make the assumption that an ion with a charge e and a mass m is placed in an air space where the field-intensity is F. Let τ_1 indicate the time interval between two consecutive collisions of the ion and a molecule, the component of the velocity of the ion in the direction of the field will then, during that time, increase by

$$\frac{e}{m} \cdot \tau_1 \cdot F.$$

The mean velocity u in the direction of and produced by the field is consequently

$$u = \tfrac{1}{2} \frac{e}{m} F \tau_1, \tag{1}$$

and the distance L covered during the time τ_1 is

$$L = u\tau_1 = \tfrac{1}{2} \frac{e}{m} F \tau_1^2. \tag{2}$$

The mean distance $\bar{L}$ traversed by the ion in the interval between two consecutive collisions is consequently

$$\bar{L} = \tfrac{1}{2} \frac{e}{m} F \overline{\tau_1^2}. \tag{3}$$

where $\overline{\tau_1^2}$ indicates the mean value of τ_1^2.

To L corresponds a mean charge displacement (*i. e.* charge times path) amounting to

$$e\bar{L} = \tfrac{1}{2} \frac{e^2}{m} \overline{\tau_1^2} F. \tag{4}$$

The mean value of τ_1 is indicated by $\bar{\tau}_1$, and the current i, which during this time would produce the same charge displacement by passing a conductor of lenght 1 cm, is determined by

$$i\bar{\tau}_1 = \tfrac{1}{2} \frac{e^2}{m} \overline{\tau_1^2} F. \tag{5}$$

Supposing the ion to be enclosed within a cube of volume one cm^3, then this

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space, owing to the presence of the ion, will acquire a conductivity σ equaling the current produced by the field-intensity $F = 1$. Consequently we have

$$\sigma\,(\mathrm{e.s.u.}) = \tfrac{1}{2}\,\frac{e^2}{m}\,\frac{\overline{\tau_1^2}}{\tau_1}. \qquad (\mathrm{e.s.u.}) \qquad (6)$$

In the above equations the use of e. s. u. has been assumed in the calculations. Expressing the charge of the ion in e. s. u., the conductivity as measured in e. m. u. will have the following value:

$$\sigma(\mathrm{e.m.u.}) = \tfrac{1}{2}\,\frac{e^2}{m}\,\frac{\overline{\tau_1^2}}{\tau_1}\cdot\frac{1}{9\cdot 10^{20}}. \qquad (e \text{ in e.s.u., } m \text{ in g., } \tau_1 \text{ in sec}) \qquad (7)$$

We shall now determine σ in a somewhat different manner, which subsequently will be used in the determination of the conductivity in the case of alternating fields.

In accordance with the assumptions made, the x-co-ordinate of the ion has to satisfy the equation:

$$m\,\frac{d^2x}{dt^2} = F\cdot e \quad \text{or} \quad \frac{dx}{dt} = \frac{e}{m}\,Ft + U_x, \qquad (8)$$

in which U_x is the component in the direction of the field of the velocity of the ion after the last collision, this direction being coincident with the X-axis.

After an interval of time τ_1, we have

$$\frac{dx}{dt} = \frac{e}{m}\,F\tau_1 + U_x,$$

and the consequent increase a in kinetic energy is

$$a = \tfrac{1}{2}m\left(U_x^2 + \frac{e^2}{m^2}F^2\tau_1^2 + 2U_x\,\frac{e}{m}\,F\tau_1\right) - \tfrac{1}{2}mU_x^2 = \tfrac{1}{2}\,\frac{e^2}{m}\,\tau_1^2F^2 + e\tau_1U_xF, \qquad (9)$$

since the components of the velocity in the other directions remain unchanged.

We assume here that the thermic velocity is high in comparison with that produced by the electric field, and we shall later give justification for this assumption. In this case positive or negative values of U_x will be equally probable and the mean increase $\bar{a}$ in kinetic energy in traversing one free path will then be

$$\bar{a} = \tfrac{1}{2}\,\frac{e^2}{m}\,\overline{\tau_1^2}F^2. \qquad (10)$$

The mean increase $\bar{A}$ per ion per second is consequently

$$\bar{A} = \frac{\bar{a}}{\tau_1} = \tfrac{1}{2}\,\frac{e^2}{m}\,\frac{\overline{\tau_1^2}}{\tau_1}\cdot F^2, \qquad (11)$$

and the corresponding conductivity is given by

$$\sigma = \frac{\bar{A}}{F^2} = \tfrac{1}{2}\,\frac{e^2}{m}\,\frac{\overline{\tau_1^2}}{\tau_1}, \qquad (12)$$

which is identical to the value found above.

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If the electric field is not constant, but has an instantaneous value f given by

$$f = F_m \cos(\omega t + \varphi),$$

then the equation (8) will be altered as follows:

$$m\frac{d^2x}{dt^2} = ef = eF_m \cos(\omega t + \varphi), \tag{13}$$

from which we derive

$$\frac{dx}{dt} = \frac{1}{\omega}\frac{e}{m} F_m (\cos\varphi \sin\omega t + \sin\varphi \cos\omega t - \sin\varphi) + U_x. \tag{14}$$

As mentioned above, we assume that the electron — and ion — velocities produced by the wave field are small in comparison with their thermic velocities. That this assumption is justified will be seen as follows:

We have for the amplitude A of the oscillations:

$$A = \frac{e}{m}\frac{1}{\omega^2} F_m, \qquad \text{(e. s. u., cm.)}$$

while the amplitude of velocity U_m is determined by:

$$U_m = \frac{e}{m}\frac{1}{\omega} F_m. \qquad \text{(e. s. u., cm sec}^{-1}\text{).}$$

Inserting here for the electron $\frac{e}{m} = 5.3 \cdot 10^{17}$ (e. s. u.) and for the field-intensity $F = 1 \cdot 10^{-5}$ Volt cm^{-1} $= \frac{1}{3} \cdot 10^{-7}$ e. s. u. cm^{-1} we get:

$$A = 1.8 \frac{10^{10}}{\omega^2}, \qquad \text{(cm)}$$

and

$$U_m = 1.8 \frac{10^{10}}{\omega}. \qquad \text{(cm sec}^{-1}\text{)}$$

The assumptions are thus justified for the conditions prevailing in radio work.

Forming from equation (14) the square, $\left(\frac{dx}{dt}\right)^2$, and taking the mean value of this term, for φ varying from $\varphi = 0$ to $\varphi = 2\pi$, we find:

$$\overline{\left(\frac{dx}{dt}\right)^2} = U_x^2 + \frac{1}{\omega^2}\frac{e^2}{m^2}(1 - \cos\omega t) F_m^2. \tag{15}$$

The mean increase a in kinetic energy per ion and per free path traversed is consequently:

$$a = \tfrac{1}{2} m \left(\overline{\left(\frac{dx}{dt}\right)^2} - U_x^2\right) = \tfrac{1}{2}\frac{1}{\omega^2}\frac{e^2}{m}(1 - \cos\omega t) F_m^2,$$

t being the time elapsed since the last collision.

At the end of the free path considered, t will be equal to τ_1, and the corresponding value of a is

$$a = \tfrac{1}{2}\frac{e^2}{\omega^2 m}(1 - \cos\omega\tau_1) F_m^2. \tag{16}$$

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Supposing the length of free path traversed to be y, then $\tau_1 = \frac{y}{U}$, where U is the mean velocity of the ion. If the mean length of the free path is equal to l, then the probability ds of y having a value between y and $y + dy$ will be given by the well known expression

$$ds = \frac{1}{l} e^{-\frac{y}{l}} \cdot dy. \tag{17}$$

The mean value $\bar{a}$ of a is then determined by

$$\bar{a} = \frac{1}{2} \frac{e^2}{\omega^2 m} F_m^2 \int_0^\infty \left[1 - \cos\left(\omega \frac{y}{U}\right)\right] \frac{1}{l} e^{-\frac{y}{l}} dy = \frac{e^2}{m} \frac{1}{\nu^2 + \omega^2} \frac{1}{2} F_m^2, \tag{18}$$

where $\nu = \frac{U}{l}$ is the average number of collisions per second performed by one ion.

The mean increase in energy per ion per second is consequently

$$\bar{A} = \nu \bar{a} = \frac{e^2}{m} \frac{\nu}{\nu^2 + \omega^2} \cdot \frac{1}{2} F_m^2, \tag{19}$$

so that for the conductivity σ corresponding to 1 ion per cubic cm we get the following expressions:

$$\sigma = \frac{\bar{A}}{\frac{1}{2} F_m^2} = \frac{e^2}{m} \frac{\nu}{\nu^2 + \omega^2} \qquad \text{(e. s. u.)} \tag{20}$$

or

$$\sigma\,[\text{e. m. u.}] = \frac{e^2}{m} \frac{\nu}{\nu^2 + \omega^2} \frac{1}{9 \cdot 10^{20}}. \qquad \text{(e in e. s. u.)} \tag{21}$$

For an electron we have $\frac{e^2}{m}$ equal to $2.53 \cdot 10^8$ and thus

$$\sigma_{[1 \text{ electron per cc.}]} = 2.81 \cdot 10^{-13} \frac{\nu}{\nu^2 + \omega^2}. \qquad \text{(e. m. u.)} \tag{22}$$

For an ion, positive or negative, we assume the mass to be 50000 times as large as that of the electron. Consequently

$$\sigma_{[1 \text{ ion per cc.}]} = 5.6 \cdot 10^{-18} \frac{\nu}{\nu^2 + \omega^2}. \qquad \text{(e. m. u.)} \tag{23}$$

For $\omega \ll \nu$ equation (20) gives

$$\sigma = \frac{e^2}{m} \frac{1}{\nu}, \qquad \text{(e. s. u.)} \tag{24}$$

while for $\omega \gg \nu$ we have

$$\sigma = \frac{e^2}{m} \frac{\nu}{\omega^2}. \qquad \text{(e. s. u.)} \tag{25}$$

Equation (20) shows that for a given frequency the conductivity is greatest at that height for which $\nu = \omega$ (For values of ν_{el} and ν_{ion} see Fig. IV. 6).

The conductivities corresponding to one electron and to one ion per cubic cm are shown in Fig. 1 for various frequencies and altitudes. It appears from the figure that at the surface of the earth and for all frequencies used in radio work one electron will contribute as much to the conductivity as about

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10^3 ions, while at altitudes exceeding 100 km and for $\omega > 10^7$, the contribution from one electron will be about the same as from 10^7 ions. For lower frequencies the difference is not quite as large.

Instead of equation (20), *G. J. Elias*[1] derives the following formula for determination of the conductivity

$$\sigma = \frac{e^2}{m} \frac{\nu}{\omega^2} \operatorname{lgn} \frac{\sqrt{\nu^2 + \omega^2}}{\nu}, \tag{26}$$

which for $\omega \ll \nu$ gives

$$\sigma = \frac{1}{2} \frac{e^2}{m} \frac{1}{\nu}, \tag{27}$$

and for $\omega \gg \nu$:

$$\sigma = \frac{e^2}{m} \frac{\nu}{\omega^2} \operatorname{lgn} \frac{\omega}{\nu}. \tag{28}$$

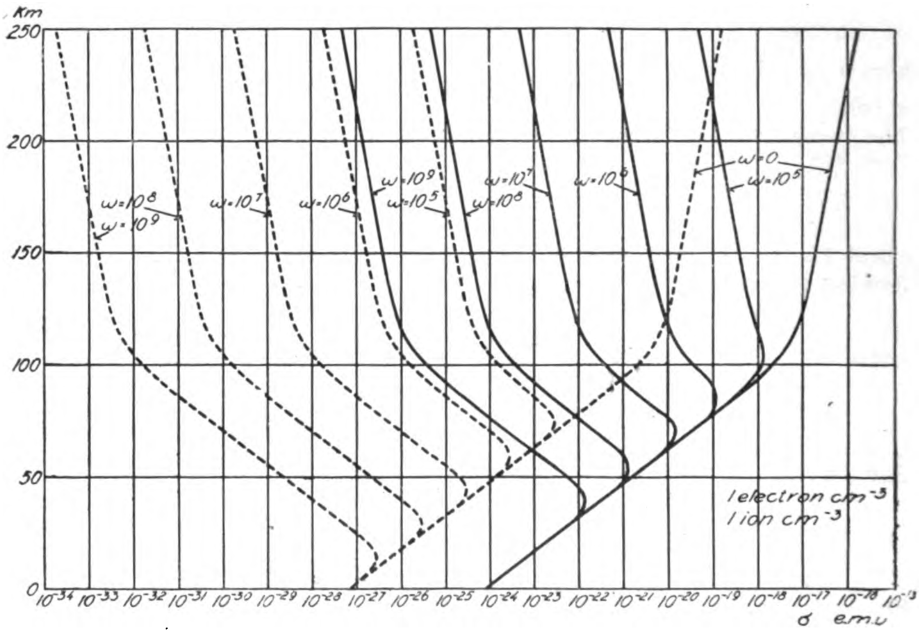

Fig. VI. 1. The curves drawn in full lines represent the Conductivity produced by one Electron per cc as a function of the Altitude *h*. The dotted curves correspond to one Ion per cc. The Composition of the Atmosphere corresponds to *F* in Chapter IV, Table 3.

Thus, for $\omega \ll \nu$ *Elias* finds only half of the conductivity value calculated by our formula while *Elias*' value for $\omega \gg \nu$ is $\operatorname{lgn} \frac{\omega}{\nu}$ times as large as ours.

The discrepancy between the formula of *Elias* and our equation (20) is explained by the different ways in which the two formulas are deduced. *Elias* assumes, in accordance with *P. Drude*[2] and a number of other authors, that the contribution to the conductivity of those ions, for which the time between two consecutive collisions is equal to τ_1, is proportional only to the mean contribution of such an ion during the time τ_1 and to the probability of τ_1

[1] *G. J. Elias:* Jahrb. d. drahtl. Telegraphie. Bd. 27, p 69. 1926.
[2] *P. Drude:* Annalen d. Physik. Bd. 1, p. 566—613. 1900.

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having a value between τ_1 and $\tau_1 + d\tau_1$, but he does not take into consideration that this contribution is proportional also to the time τ_1 itself.

It is easily seen that our method of reasoning will give a higher conductivity than the one found according to *Drude*, when the time interval between two collisions is much smaller than the period of the waves, inasmuch as it gives an increased weight to cases for which τ_1 exceeds its mean value, and just these cases make a relatively large contribution to the conductivity, In the other extreme case, when the period is much shorter than the time between two collisions, the cases for which τ_1 has a relatively high value will, on the contrary, contribute relatively little to the conductivity, and by giving these cases an increased weight a lower value for the conductivity will consequently be obtained.

The objection here made to *Drude*'s method of calculating the conductivity — the method followed by *Elias* and others — does not apply to the method used by *H. A. Lorentz*[1] and a number of later authors in determining the conductivity of metals on the basis of the electron theory.

In our deductions we have assumed that the electrons and ions always move at their mean velocity. This is undoubtedly not the case. *H. A. Lorentz* assumes that the velocities of the electrons distribute themselves in accordance with *Maxwell*'s law of distribution, and in calculating the conductivity of metals this is a very natural assumption. But it would not be entirely correct here, at any rate not as far as the electrons are concerned, since we make the presumably justified assumption that the electrons are not in complete thermic equilibrium with the atmosphere, but have somewhat higher velocities.

Since we do not know the actual law of distribution for the velocity of the electrons, and since we presumably introduce only a slight error by counting on constant velocities for the electrons as well as for the ions, we have chosen the simple and generally followed method of assuming the velocity to be constant.

It has been necessary to enter somewhat further into the question of conductivity, because in most treatments of the subject only more or less correct values for the conductivity have been given for the special cases where either $v \gg \omega$, corresponding to conditions near the surface of the earth, or $v \ll \omega$, which for $\omega > 10^7$ corresponds to conditions of the atmosphere at high altitudes. But a glance at Fig. 1 shows that none of these approximations are at all suitable for representing the conductivity within the wide ranges of altitude and frequency here concerned. In addition, it may be mentioned that quite misleading representations of conditions have appeared in recent authoritative publications.

An exception is formed by the important paper by *G. J. Elias* who gives a general determination of the conductivity valid throughout the entire range. But since our investigations on this point were finished before the appearance of *Elias*' paper, and since we, as mentioned above, arrive at a somewhat different result than this author, we have thought it proper to publish our own researches also.

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[1] *H. A. Lorentz:* The Theory of Electrons. Notes 29. 2nd Ed. (Leipzig. 1916).

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2. *The Influence of Electrons and Ions on the Dielectric Constant of the Air at Various Heights.*

For ions actuated by the field $f = F_m \cos(\omega t + \varphi)$ we find according to equation (14) the following velocity in the direction of the field:

$$\frac{dx}{dt} = \frac{1}{\omega}\frac{e}{m} F_m \left(\sin(\omega t + \varphi) - \sin\varphi\right) + U_x \tag{14}$$

giving $\frac{dx}{dt}$ equal to U_x for $t = 0$.

We shall next find the mean amplitude $\overline{\Phi(t)}$ of the velocity component in phase with $\sin(\omega t + \varphi)$, *i. e.* $\frac{\pi}{2}$ in phase behind the acting force. For this purpose we write preliminarily

$$\Psi(t) = 2\int_0^t \frac{dx}{dt}\sin(\omega t + \varphi)\,dt = -2\frac{U_x}{\omega}\left(\cos(\omega t + \varphi) - \cos\varphi\right)$$

$$+\frac{2}{\omega}\frac{e}{m}F_m\left(\frac{1}{2}t - \frac{1}{4\omega}\sin 2(\omega t+\varphi) + \frac{1}{\omega}\sin\varphi\cos(\omega t+\varphi) + \frac{1}{4\omega}\sin 2\varphi - \frac{1}{\omega}\sin\varphi\cos\varphi\right).$$

The mean value $\Phi(t)$ of this expression for φ varying from $\varphi = 0$ to $\varphi = 2\pi$, is determined by

$$\Phi(t) = \frac{1}{2\pi}\int_0^{2\pi}\Psi(t)\,d\varphi = \frac{1}{\omega}\frac{e}{m}\left(t - \frac{1}{\omega}\sin\omega t\right)F_m. \tag{29}$$

The mean amplitude of the component which is $\frac{\pi}{2}$ in phase behind the acting force is consequently, for the free paths traversed during the time t, determined by

$$\overline{\Phi(t)} = \frac{1}{t}\,\Phi(t) = \frac{e}{\omega m}\left(1 - \frac{1}{\omega t}\sin\omega t\right)F_m. \tag{30}$$

If the free path corresponding to t is equal to y, then t will be equal to $\frac{y}{U}$. The probability ds of the free path having a value between y and $y + dy$ is, as mentioned above,

$$ds = \frac{1}{l}e^{-\frac{y}{l}}dy,$$

where l is the mean value of the free path.

The resultant mean value $\bar{b}$ of the amplitude of the velocity component $\frac{\pi}{2}$ in phase behind the acting force will consequently be:

$$\bar{b} = \int_{y=0}^{y=\infty}\overline{\Phi(t)}\,\frac{y}{l}\,ds = \frac{e}{\omega m}F_m\int_0^{\infty}\left[1 - \frac{U}{\omega y}\sin\left(\omega\frac{y}{U}\right)\right]\frac{y}{l^2}e^{-\frac{y}{l}}dy = \frac{e}{m}\frac{\omega}{\nu^2+\omega^2}F_m, \tag{31}$$

the factor $\frac{y}{l}$ indicating the relative duration of the condition corresponding to $\overline{\Phi\left(\frac{y}{U}\right)}$, and ν being equal to $\frac{U}{l}$ as usual.

In order to illustrate the meaning of $\bar{b}$ we shall consider the following simple case, namely electrons or ions moving quite freely without collisions.

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We may then write

$$\frac{d^2x}{dt^2} = \frac{e}{m} F_m \cos \omega t \quad \text{and} \quad \frac{dx}{dt} = \frac{e}{\omega m} F_m \sin \omega t. \qquad \text{(e. s. u.).} \qquad (32)$$

The moving ion corresponds to a current element of length 1 cm and carrying a current i of the following value

$$i = e\frac{dx}{dt} = \frac{e^2}{\omega m} F_m \sin \omega t = \frac{1}{\omega L_0} F_m \sin \omega t, \qquad \text{(e. s. u.).} \qquad (33)$$

where $L_0 = \frac{m}{e^2}$ corresponds to a self-inductance.

We consider further (see Fig. 2 III) a condenser with a plate area of 1 cm²

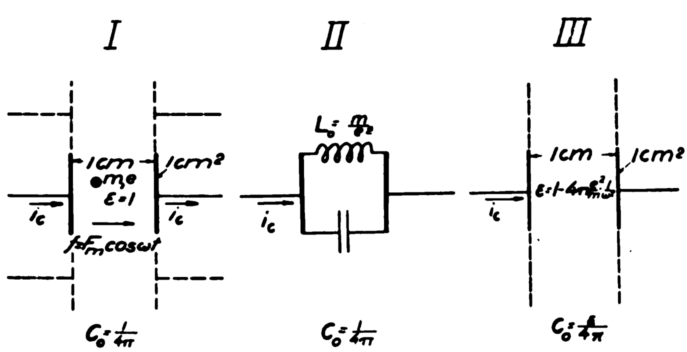

Fig. VI. 2. The Condenser indicated by full lines is considered as an element of a larger condenser, so that its capacity without any ion may be expressed by $C = \frac{1}{4\pi}$. II is equivalent to I with one ion. III again is equivalent to II when the Dielectric Constant of the medium is $\varepsilon = 1 - \Delta\varepsilon = 1 - 4\pi \frac{e^2}{m} \cdot \frac{1}{\omega^2}$.

and a plate distance of 1 cm. Across this condenser is impressed the voltage $f = F_m \cos \omega t$. The charging and displacement current i_c is then determined by

$$i_c = C\frac{df}{dt} = -\omega C F_m \sin \omega t = -\omega \frac{\varepsilon}{4\pi} F_m \sin \omega t, \qquad \text{(e. s. u.).} \qquad (34)$$

C being equal to $\frac{\varepsilon}{4\pi}$, where ε is the dielectric constant of the medium.

The unit condenser shown in Fig. 2 I and with one ion is consequently equivalent to the condenser $C_0 = \frac{1}{4\pi}$ shown in Fig. 2 II when shunted by the self-inductance $L_0 = \frac{m}{e^2}$. This circuit again is equivalent to the unity condenser shown in III when the dielectric constant ε of the medium is determined by

$$\varepsilon = 1 - \Delta\varepsilon = 1 - 4\pi \frac{e^2}{m} \cdot \frac{1}{\omega^2}. \qquad \text{(e. s. u).} \qquad (35)$$

Similarly from the velocity amplitude $\bar{b}$ we get a corresponding current amplitude $e\bar{b}$ and a comparison between (31) and (34) shows that the reduction of the dielectric constant of the air caused by one ion per cc is determined by

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$$\Delta\varepsilon = 4\pi \frac{e^2}{m} \frac{1}{\nu^2 + \omega^2}. \qquad \text{(e. s. u.)} \qquad (36)$$

For the electrons we have $4\pi \frac{e^2}{m} = 3.18 \cdot 10^9$, and for an ion 50 000 times as heavy we have $4\pi \frac{e^2}{m} \cong 6.4 \cdot 10^4$.

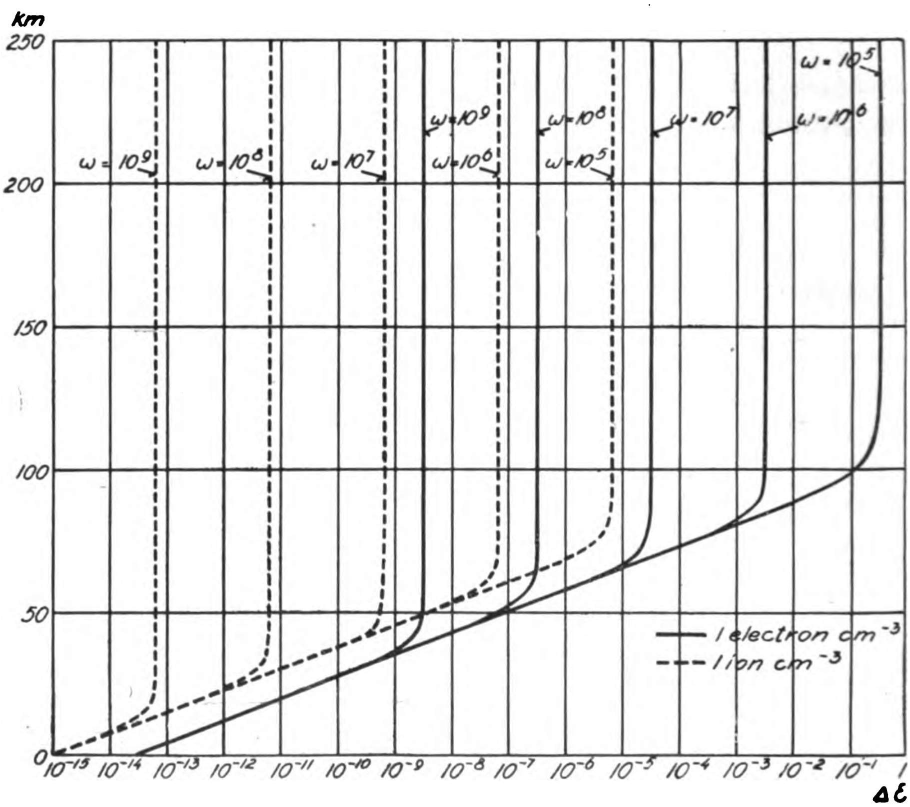

Fig. VI. 3. The fully drawn curves show the value of $\Delta\varepsilon_{el}$, the dotted ones the value of $\Delta\varepsilon_{ion}$. The Composition of the Atmosphere corresponds to F in Table 3 Chapter IV.

For $\omega \ll \nu$ we find

$$\Delta\varepsilon = 4\pi \frac{e^2}{m} \frac{1}{\nu^2} \qquad (37)$$

and for $\omega \gg \nu$

$$\Delta\varepsilon = 4\pi \frac{e^2}{m} \frac{1}{\omega^2}. \qquad (38)$$

The values of $\Delta\varepsilon$ corresponding to one electron and to one ion per cubic cm are shown in Fig. 3 for various frequencies and altitudes. It will be noted that $\Delta\varepsilon_{el}$ does not differ much from $\Delta\varepsilon_{ion}$ for $\omega < 10^7$, and for heights below 40 km. For high frequencies and high altitudes, on the contrary, we get

$$\Delta\varepsilon_{el} = 5 \cdot 10^4 \cdot \Delta\varepsilon_{ion}.$$

Instead of equation (36) *G. J. Elias* finds the following formula

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$$\Delta\varepsilon = 4\pi \frac{e^2}{m} \frac{v}{\omega^3} \left(\frac{\omega}{v} - \frac{\pi}{2} + \operatorname{arc\,tg} \frac{v}{\omega} \right) \tag{39}$$

giving for $\omega \ll v$

$$\Delta\varepsilon = \frac{4\pi}{3} \frac{e^2}{m} \frac{1}{v^2}. \tag{40}$$

This value is therefore one third of ours.

For $\omega \gg v$ (39) is reduced to:

$$\Delta\varepsilon = 4\pi \frac{e^2}{m} \frac{1}{\omega^2}, \tag{41}$$

i. e. the same value as found by us[1].

For the reason mentioned before under the determination of the conductivity there is again a difference between the formula for $\Delta\varepsilon$ as presented by *Elias* and by ourselves. When the two methods of calculation in this case lead to the same result for $\omega \gg v$, the reason is that as the time τ_1 between two consecutive collisions becomes great in comparison with the period of the waves, $\Delta\varepsilon$ approaches a value which is independent of this time.

W. H. Eccles[2] is probably the first to have called attention to the reduction of the dielectric constant due to the free ions in the atmosphere, and he has at the same time given a substantially correct formula for the calculation of this reduction. However, in calculating the very few numerical values for which he uses this formula, he has with one single exception assumed ions much heavier ($m \cong 2.2 \cdot 10^6$ times the mass of the electron) than those used in our calculations. His results are therefore of less direct importance to the treament of the present problem. Sir *Joseph Larmor*[3] has called attention to *Eccles'* work, and has at the same time deduced formula (38) which is valid for $\omega \gg v$, and in explaining the propagation of waves he has made a series of interesting observations concerning the use of this formula[4].

3. *The Ratio between the Conductivity and the Reduction in the Dielectric Constant caused by the free Ions and Electrons.*

According to equations (21) and (36) the ratio is

$$\frac{\sigma}{\Delta\varepsilon} = \frac{v}{4\pi c^2} = 8.84 \cdot 10^{-23}\, v \tag{42}$$

where, as previously, σ is measured in e. m. u. and $\Delta\varepsilon$ in e. s. u., and on

[1] *Elias* writes the formula (40) as $\Delta\varepsilon = 2\pi \frac{e^2}{m} \frac{1}{v^2}$, but this must presumably be due to a clerical error.

[2] *W. H. Eccles:* Proc. Roy. Soc. (A), Vol. 87, p. 79—99. 1912.

[3] *J. Larmor:* Phil. Mag. (6). Vol. 48, p. 1025—1036. 1924.

[4] Very recently other authors have deduced these correct formulas for σ and $\Delta\varepsilon$ in another manner. The formulas have thus been proved by *H. Lassen* in a very simple way (Jahrb. d. drahtl. Tel. Bd. 28, p. 139—147. Novbr. 1926) and in a very interesting manner by *W. G. Baker* and *C. W. Rice* (Refraction of Short Radio Waves in the Upper Atmosphere, Complete Paper. New York 1926. Abridgement Journ. A. I. E. E. p. 535—537. 1926).

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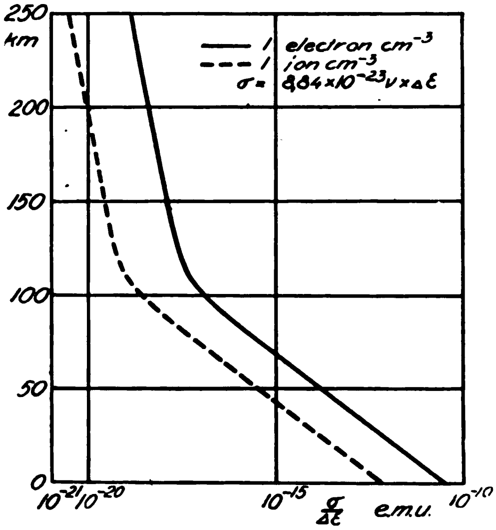

Fig. VI. 4. The Ratio between the Conductivity σ and the Reduction $\Delta\varepsilon$ of the Dielectric Constant. The full-line curve applies to electrons, the dotted one to ions. The distribution of atmospheric pressure on which the determination of the number of collisions per second ν has been based is F.

the assumption that only one kind of ions has appreciable influence. In this case the ratio is dependent only on the number of collisions of the ions concerned.

If, on the other hand, there are several kinds of ions, then

$$\sigma = \frac{1}{c^2} \sum \frac{e_1^2}{m_1} \frac{\nu_1}{\nu_1^2 + \omega^2}, \tag{43}$$

and

$$\Delta\varepsilon = 4\pi \sum \frac{e_1^2}{m_1} \frac{1}{\nu_1^2 + \omega^2}, \tag{44}$$

whence

$$\frac{\sigma}{\Delta\varepsilon} = \frac{1}{4\pi c^2} \frac{\sum \frac{e_1^2}{m_1} \frac{\nu_1}{\nu_1^2 + \omega^2}}{\sum \frac{e_1^2}{m_1} \frac{1}{\nu_1^2 + \omega^2}}. \tag{45}$$

Fig. 4 shows the values of this ratio, for electrons as well as for ions, for the atmosphere F.

4. *Concluding Remarks.*

In order to facilitate a review of the results previously deduced, and their application to various problems we shall make the following remarks:

Fig. 5 shows a condenser with terminals a and b and with a plate area of A sq cm and a plate distance of b cm, the dielectric constant of the dielectric medium being $\varepsilon_{\text{e.s.u.}}$ and the conductivity $\sigma_{\text{e.m.u.}}$. This condenser will be equivalent to the circuits II and III as shown, when the constants R, C and L are so chosen that

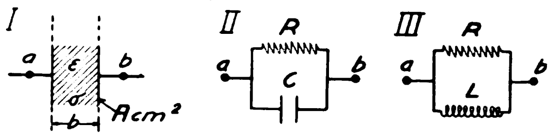

Fig. VI. 5. The three circuits shown are equivalent, provided that the constants R, C and L be determined in accordance with equations (46) and (47).

$$R = \frac{b}{A\sigma} \cdot 10^{-9} \qquad (\sigma \text{ in e. m. u., } R \text{ in ohms}) \tag{46}$$

and

$$\left.\begin{matrix} \omega C \\ -\dfrac{1}{\omega L} \end{matrix}\right\} = \frac{\omega\varepsilon A}{4\pi b} \cdot \frac{1}{9 \cdot 10^{11}}, \qquad (\varepsilon \text{ in e. s. u., } C \text{ in farads, } L \text{ in henrys, } \omega C \text{ in ohm}^{-1}) \tag{47}$$

where for $\varepsilon > 0$ the equivalent circuit is as shown in II with R and C in parallel, while for $\varepsilon < 0$ the circuit III is used with R and L in parallel.

For N ions per c c we have according to (36) and (21):

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$$\varepsilon = 1 - N \cdot 4\pi \frac{e^2}{m} \cdot \frac{1}{\omega^2 + \nu^2} = 1 - k \qquad \text{(e. s. u.)} \qquad (48)$$

and

$$\sigma = N \cdot \frac{e^2}{mc^2} \cdot \frac{\nu}{\omega^2 + \nu^2} = \frac{\nu}{4\pi c^2} \cdot k, \qquad \text{(e. m. u.)} \qquad (49)$$

where

$$k = N \cdot 4\pi \frac{e^2}{m} \cdot \frac{1}{\omega^2 + \nu^2}. \qquad (50)$$

Inserting these values in (46) and (47) we find:

$$R = \frac{4\pi b}{\nu k A} \cdot 9 \cdot 10^{11}, \qquad \text{(ohms)} \qquad (51)$$

and

$$\left.\begin{matrix} \omega C \\ -\frac{1}{\omega L} \end{matrix}\right\} = \frac{\omega(1-k)A}{4\pi b} \cdot \frac{1}{9 \cdot 10^{11}} \qquad (\text{ohm}^{-1}) \qquad (52)$$

or

$$\left.\begin{matrix} \frac{1}{\omega C} \\ -\omega L \end{matrix}\right\} = \frac{4\pi b}{\omega(1-k)A} \cdot 9 \cdot 10^{11}. \qquad \text{(ohms)} \qquad (53)$$

For $0 \leqq k \leqq 1$ we consequently have $\omega CR = \frac{\omega}{\nu} \cdot \frac{1-k}{k}$, and for $k \geqq 1$ we have

$$\frac{R}{\omega L} = \frac{\omega}{\nu} \cdot \frac{k-1}{k}. \qquad (54)$$

In Fig. 6 we have shown the quantities ωC, $\frac{1}{\omega L}$ and $\frac{1}{R}$ as functions of k.

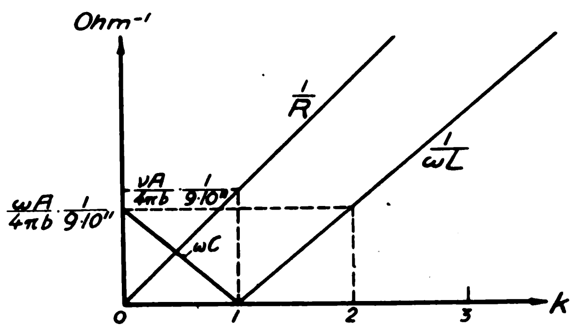

Fig. VI. 6. ωC, $\frac{1}{\omega L}$ and $\frac{1}{R}$ as functions of k.

We shall now consider the quite complicated oscillating circuit shown in Fig. 7 I, consisting of an inductance L_0 a resistance R_0 and a capacity combination consisting of an ideal condenser C_0 shunted by a similarly ideal condenser C_2 in series with a condenser T, the dielectric medium of the latter having a dielectric constant of $\varepsilon_{e.m.u.}$ and a conductivity $\sigma_{e.m.u.}$. We shall attempt to determine the wave length λ of the circuit shown for various values of ε and σ, these constants being determined by means of the equations (48) and (49). The individual cases II to VI will now be considered.

Case II. $k = 0$; $\varepsilon = 1$ and $\sigma = 0$. The resultant capacity will here be

$$C_{II} = C_0 + \frac{C_1 C_2}{C_1 + C_2} \quad \text{and} \quad \lambda = 2\pi \sqrt{L_0 C_{II}}. \qquad (56)$$

Since the loss in the condenser is zero in this case, the expression $\frac{1}{R_{eff}^2}$ will assume its maximum value, R_{eff} being the effective resistance of the entire oscillating circuit and in this case therefore equal to the effective resistance

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R_0 of the inductance coil. This condition corresponds to point II in Fig. 8, which shows the wave length of the oscillating circuit as a function of $\frac{1}{R^2_{\text{eff}}}$.

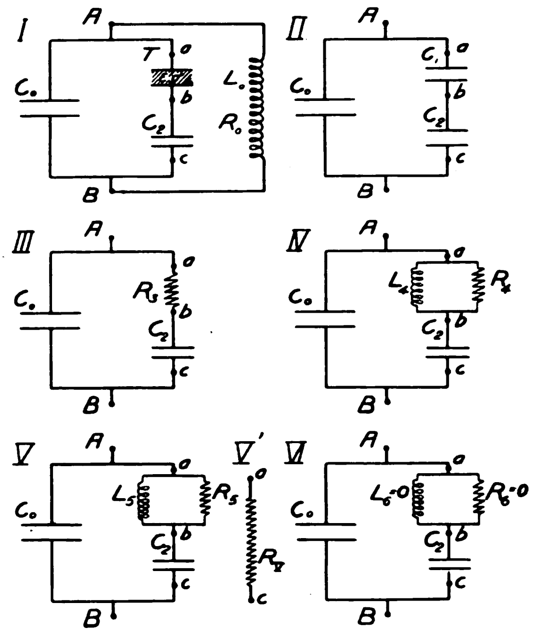

Fig. VI. 7. Figures II to VI show different states of the oscillating circuit shown in I for different values of ε and σ of the dielectric medium in the condenser T; II corresponds to $k=0$ or $\varepsilon=1$ and $\sigma=0$; III to $k=1$ or $\varepsilon=0$; IV to $\varepsilon<0$; V to resonance of the circuit between a and c and VI to $k \rightarrow \infty$.

Case III. $k=1$; $\varepsilon \rightarrow 0$ and $\sigma=\frac{v}{4\pi c^2}$, $R_3=\frac{4\pi b}{vA}\cdot 9\cdot 10^{11}$.
We find

$$\frac{1}{Z_{AB}}=\frac{\omega^2 C_2^2 R_3+j\omega C_0\left(\omega^2 C_2^2 R_3^2+1+\frac{C_2}{C_0}\right)}{1+\omega^2 C_2^2 R_3^2}=\frac{1}{R_{III}}+j\omega C_{III}, \tag{57}$$

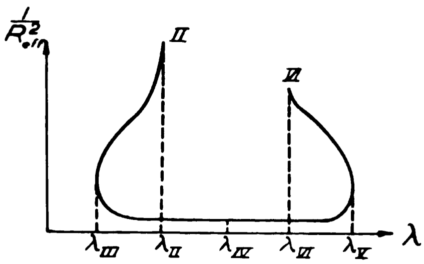

Fig. VI. 8. Corresponding values of $\frac{1}{R^2_{\text{eff}}}$ and λ for various values of k.

where Z_{AB} is the impedance of the circuit between A and B, and where

$$\left.\begin{aligned} R_{III}&=R_3+\frac{1}{\omega^2 C_2^2 R_3} \\ \text{and} \qquad C_{III}&=C_0+\frac{C_2}{1+\omega^2 C_2^2 R_3^2}. \end{aligned}\right\} \tag{58}$$

For $\omega^2 C_1 C_2 R_3^2 \gtreqless 1$ we have $C_{III} \lesseqgtr C_{II}$, and for sufficiently great values of R_3 we have $\lambda_{III}<\lambda_{II}$, as assumed in Fig. 8. The corresponding point on the

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$\left(\lambda, \frac{1}{R_{\text{eff}}}\right)$-curve must be situated below the point II, since in this case we have a greater attenuation, the condenser C_{III} (and the inductance L_0) being shunted by the resistance R_{III}.

Case IV. $k > 1$, $\varepsilon < 0$. Here $\lambda_{\text{IV}} > \lambda_{\text{III}}$ and $R_{\text{eff IV}} > R_{\text{eff III}}$.

Cases V and VI. The last mentioned case occurs, when k has become so great that both $R_6 \rightarrow 0$ and $L_6 \rightarrow 0$. In that case we have

$$\lambda_{\text{VI}} = 2\pi \sqrt{L_0(C_0 + C_2)}. \tag{59}$$

In case V k must have such a value that the circuit between a and c will be in resonance for the frequency considered, so that this circuit may be replaced by the resistance R_{V}. The impedance of the circuit between a and c is without difficulty found to be

$$Z_{ac} = \frac{\omega^3 C_2 L_5^2 R_5 - j\,(\omega^2 L_5^2 + R_5^2 - \omega^2 L_5 C_2 R_5^2)}{\omega C_2 (R_5^2 + \omega^2 L_5^2)}. \tag{60}$$

For $R_5^2 + \omega^2 L_5^2 = \omega^2 L_5 C_2 R_5^2$ the circuit ac is in resonance, and we have

$$Z_{ac} = R_{\text{V}} = \frac{\omega^2 L_5^2 R_5}{R_5^2 + \omega^2 L_5^2} = \frac{L_5}{R_5 C_2}. \tag{61}$$

For the oscillating circuit formed by L_0, C_0 and R_{V} the wave length λ_{V} is

$$\lambda_{\text{V}} = \frac{2\pi}{\sqrt{\frac{1}{L_0 C_0} - \frac{1}{(C_0 R_{\text{V}})^2}}}, \tag{62}$$

the small resistance R_0 of the inductance coil being disregarded.

It will easily be seen that for

$$R_{\text{V}}^2 \gtreqless L_0 \frac{C_0 + C_2}{C_0 C_2} \quad \text{we have } \lambda_{\text{V}} \lesseqgtr \lambda_{\text{VI}}. \tag{63}$$

The value of k corresponding to $R_5^2 + \omega^2 L_5^2 = \omega^2 L_5 C_2 R_5^2$ is determined by

$$k_{\text{V}} = \frac{\omega^2 L_5 C_2 - 1 \pm \frac{v}{\omega} \sqrt{\omega^2 L_5 C_2 - 1}}{\omega^2 L_5 C_2 - 1 - \frac{v^2}{\omega^2}}. \tag{64}$$

For $\omega^2 L_5 C_2 \geqq 1$ a value of k may be found for which the impedance of the circuit ac becomes equal to a simple resistance R_{V}.

It thus appears that the $\left(\lambda, \frac{1}{R_{\text{eff}}^2}\right)$ curve may assume the shape shown in Fig. 8 where $\lambda_{\text{III}} < \lambda_{\text{II}}$ and $\lambda_{\text{V}} > \lambda_{\text{VI}}$.

We have used the above example because *H. Gutton* and *J. Clément*[1] have examined the wave lengths of the oscillating circuit shown in Fig. 9, the degree

[1] *H. Gutton* et *J. Clément:* L'Onde électrique. Vol. 6, p. 137—151. April 1927.

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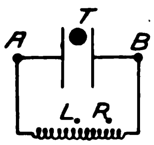

Fig. VI. 9. Oscillating circuit used by *Gutton* and *Clément*. T is the ionization tube.

of ionization in the evacuated glass tube T being varied in suitable manner. It will be seen that this arrangement is perfectly analogous to the one shown in Fig. 7 I. The experiments of *Gutton* and *Clément* resulted in a curve corresponding exactly to the one shown in Fig. 8. We have desired to draw attention to this interesting experimental work by *Gutton* and *Clément*, which otherwise falls somewhat outside the scope of the present publication, because in our opinion these authors draw some erroneous conclusions from their experimental results, and these conclusions, unless contradicted at once, might lead to a lack of confidence in our values of $\Delta\varepsilon$ and σ. The authors write as follows.

»Lorsqu'on augmente l'ionisation, la constante diélectrique cesse de décroître et passe, ... par une valeur minimum. Pour des ionisations plus fortes elle augmente très vite et devient plus grande que celle du gaz non ionisé. Le résonateur est alors très amorti.

En augmentant encore l'ionisation, la constante diélectrique passe par une valeur maximum qui correspond à la pointe gauche de la courbe. L'amortissement diminue et la courbe tend vers un point qui correspondrait à une conductibilité parfaite du gaz, pour laquelle le champ électromagnetique ne pénètre plus dans le milieu ionisé et pour laquelle, par suite, l'amortissement n'est plus augmenté.

La diminution de la constante diélectrique prévue par la théorie de *Eccles* ne se continue donc pas constamment lorsque l'ionisation augmente. Elle atteint des valeurs pour lesquelles il n'est plus possible de négliger les actions mutuelles des ions.

L'expérience montre qu'il se produit alors une augmentation rapide de la constante diélectrique, augmentation qui correspond nécessairement à une inversion de la différence de phase entre l'élongation des oscillations des ions et celle des oscillations de la force électrique de telle sorte que le courant de convection, au lieu de produire une diminution apparente de la constante diélectrique, provoque, au contraire, une augmentation apparente de celle-ci.

Les phénomènes que nous venons de décrire s'expliquent immédiatement en admettant que les ions ne sont pas libres et qu'il faut introduire, outre les actions analogues à des frottements qui resultent des chocs, des forces quasi élastiques qui proviennent des attractions mutuelles entre les ions et des mouvements qu'elles produisent.«[1]

We have seen above that a correct use our values of $\Delta\varepsilon$ and σ fully explain the experimental results of *Gutton* and *Clément*. The disagreement with »*Eccles*' theory« found by these authors is merely due to the fact that the collisions between the ions (electrons) and the molecules have been disregarded in that theory by assuming $\Delta\varepsilon = N \cdot 4\pi \frac{e^2}{m\omega^2}$ and $\sigma = 0$. This is of course not justifiable, but there is no reason at all for leaving the simple basis on which the above theory has been built with respect to the values of the dielectric constant and the conductivity of an ionized gas.

[1] l. c., p. 146.

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CHAPTER VII.

THE INFLUENCE OF THE EARTH'S MAGNETIC FIELD ON THE PROPAGATION OF RADIO WAVES.

1. *The Movement of an Ion in a Magnetic Field when Influenced by a Radio Wave.*

Certain authors[1] have attempted to explain the low reaching ability of waves with a wave length around 200 meters as well as various other characteristic features of the propagation of waves, as being due to the influence exerted by the earth's magnetic field on the shape of the paths traversed by the electrons in the atmosphere when acted upon by the electric field of the waves. It will therefore be necessary to enter somewhat further into this question.

For this purpose we consider an ion, with charge e and mass m, placed in an electric field (E_x, E_y, E_z) and a magnetic field (H_x, H_y, H_z). The motion of the particle is then determined by the equations:

$$\left.\begin{aligned} \frac{d^2x}{dt^2} &= \mu\left(E_x + \frac{1}{c}\left(H_z\frac{dy}{dt} - H_y\frac{dz}{dt}\right)\right), \\ \frac{d^2y}{dt^2} &= \mu\left(E_y + \frac{1}{c}\left(H_x\frac{dz}{dt} - H_z\frac{dx}{dt}\right)\right), \\ \frac{d^2z}{dt^2} &= \mu\left(E_z + \frac{1}{c}\left(H_y\frac{dx}{dt} - H_x\frac{dy}{dt}\right)\right). \end{aligned}\right| \quad \begin{aligned} &\mu = \frac{e}{m} \\ &\left[\begin{aligned} &E \text{ and } e \text{ in e. s. u.,} \\ &H \text{ in gauss.} \end{aligned}\right] \\ &c = 3\cdot 10^{10}\,\text{cm sec}^{-1} \end{aligned} \tag{I}$$

In the following we assume that the magnetic field is constant and homogeneous, that is:

$$H_x = H_{x0}, \qquad H_y = H_{y0}, \qquad H_z = H_{z0},$$

and that the electric field varies harmonically in the direction of the Z-axis:

$$E_x = 0; \qquad E_y = 0; \qquad E_z = E_0 \cos(\omega t + \varphi).$$

We disregard here the magnetic field of the electro-magnetic waves themselves, since the intensity of this field may be considered negligible in comparison with the intensity of the constant magnetic field, and the ratio between the two forces with which the magnetic field and the electric field of the waves act upon an ion is equal to $\frac{u}{c}$, where u and c, respectively, are the velocities of the ion and of light. Since we here consider only relatively slowly moving ions we may without any appreciable error disregard the influence of the magnetic field of the waves themselves.

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[1] *E. V. Appleton* and *M. A. F. Barnett:* »Electrician«, April 3rd, 1925, p. 398; Proc. Camb. Phil. Soc. Vol. 22, Part 5, p. 672—675, 1925. *H. W. Nichols* and *J. C. Schelleng:* The Bell System Technical Journal, Vol. 4, No. 2, April 1925.

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We further introduce the following abbreviations

$$h_x = \frac{\mu}{c} H_{x0} = \frac{e}{mc} H_{x0}; \quad h_y = \frac{\mu}{c} H_{y0}; \quad h_z = \frac{\mu}{c} H_{z0}; \quad h = +\sqrt{h_x^2 + h_y^2 + h_z^2}. \tag{II}$$

The equations (I) may then be reduced to:

$$\left.\begin{aligned} \frac{d^2x}{dt^2} &= h_z \frac{dy}{dt} - h_y \frac{dz}{dt}, \\ \frac{d^2y}{dt^2} &= h_x \frac{dz}{dt} - h_z \frac{dx}{dt}, \\ \frac{d^2z}{dt^2} &= \mu E_0 \cos(\omega t + \varphi) + h_y \frac{dx}{dt} - h_x \frac{dy}{dt}. \end{aligned}\right\} \tag{I'}$$

Elimination of y and z from the above equations results in:

$$\frac{d^4x}{dt^4} + h^2 \frac{d^2x}{dt^2} = h_x h_z \mu E_0 \cos(\omega t + \varphi) + h_y \omega \mu E_0 \sin(\omega t + \varphi). \tag{1}$$

This equation is satisfied by the particular solution

$$x = \frac{h_x h_z \mu E_0}{\omega^2(\omega^2 - h^2)} \cos(\omega t + \varphi) + \frac{h_y \omega \mu E_0}{\omega^2(\omega^2 - h^2)} \sin(\omega t + \varphi), \tag{2}$$

which after insertion into (I') gives corresponding solutions for y and z, namely:

$$y = \frac{h_y h_z \mu E_0}{\omega^2(\omega^2 - h^2)} \cos(\omega t + \varphi) - \frac{h_x \omega \mu E_0}{\omega^2(\omega^2 - h^2)} \sin(\omega t + \varphi). \tag{3}$$

and

$$z = \frac{(h_z^2 - \omega^2) \mu E_0}{\omega^2(\omega^2 - h^2)} \cos(\omega t + \varphi). \tag{4}$$

The solutions (2), (3) and (4) represent the oscillations of the ion as forced by the electric alternating field.

In order to obtain the complete solution of (1) we must, however, also consider the free oscillations of the ion in the magnetic field. We therefore write $E_0=0$, whereby the right-hand side of equation (1) becomes equal to zero, and this equation will then be satisfied by:

$$x = A_1 \cos(ht + \varphi_1) + B_1 t + C_1, \tag{2a}$$

and correspondingly for y and z:

$$y = A_2 \cos(ht + \varphi_2) + B_2 t + C_2, \tag{3a}$$

and

$$z = A_3 \cos(ht + \varphi_3) + B_3 t + C_3. \tag{4a}$$

The solutions (2a), (3a) and (4a) represent the free oscillations performed by the ion in the magnetic field, when the ion is not actuated by any electric field. The ion will then, as it is well known, move in a helical path having for its axis the direction of the magnetic field. The radius of the helix is

$$r = \frac{U_n}{h}, \tag{2b}$$

U_n being the velocity of the ion perpendicular to the direction of the magnetic field.

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The pitch s of the helix is

$$s = 2\pi \frac{U_t}{h}, \tag{3b}$$

where U_t is the component of the velocity of the ion in the direction of the magnetic field. Finally the time of revolution T is determined by:

$$T = \frac{2\pi}{h}. \tag{4b}$$

The angular frequency is consequently equal to h.

The time of revolution — and frequency — thus depends only on the intensity of the magnetic field and on the ratio between the charge and the mass of the ion.

Substituting for x, y and z in the equations (I') the expressions (2a), (3a) and (4a) we find, as $E = 0$, that the arbitrary constants A, B and φ have to satisfy the conditions:

$$\frac{B_1}{h_x} = \frac{B_2}{h_y} = \frac{B_3}{h_z}; \tag{5}$$

$$\left.\begin{aligned}
A_1 h \cos\varphi_1 &= A_2 h_z \sin\varphi_2 - A_3 h_y \sin\varphi_3,\\
A_1 h \sin\varphi_1 &= -A_2 h_z \cos\varphi_2 + A_3 h_y \cos\varphi_3,\\
A_2 h \cos\varphi_2 &= A_3 h_x \sin\varphi_3 - A_1 h_z \sin\varphi_1,\\
A_2 h \sin\varphi_2 &= -A_3 h_x \cos\varphi_3 + A_1 h_z \cos\varphi_1,\\
A_3 h \cos\varphi_3 &= A_1 h_y \sin\varphi_1 - A_2 h_x \sin\varphi_2,\\
A_3 h \sin\varphi_3 &= -A_1 h_y \cos\varphi_1 + A_2 h_x \cos\varphi_2.
\end{aligned}\right\} \tag{6}$$

The equations (6) are satisfied by A_1, A_2, A_3, φ_1, φ_2 and φ_3 when these constants satisfy the following equations:

$$\frac{A_1}{\sqrt{h_y^2 + h_z^2}} = \frac{A_2}{\sqrt{h_x^2 + h_z^2}} = \frac{A_3}{\sqrt{h_x^2 + h_y^2}}, \tag{7}$$

$$\left.\begin{aligned}
\cos(\varphi_1 - \varphi_2) &= -\frac{h_x h_y}{\sqrt{(h_z^2+h_x^2)(h_z^2+h_y^2)}}, \quad \cos(\varphi_2 - \varphi_3) = -\frac{h_y h_z}{\sqrt{(h_x^2+h_y^2)(h_x^2+h_z^2)}},\\
\cos(\varphi_1 - \varphi_3) &= -\frac{h_x h_z}{\sqrt{(h_y^2+h_x^2)(h_y^2+h_z^2)}}.
\end{aligned}\right\} \tag{8}$$

The initial state of the particle is determined by its three co-ordinates and three components of velocity for the time $t = 0$.

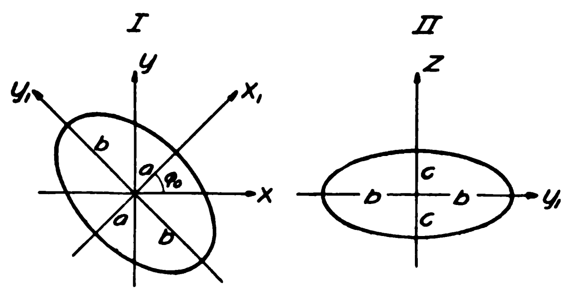

Fig. VII. 1. I. The System of Co-ordinates is rotated around the Z-axis into the position(X_1,Y_1), and the direction of the X_1-axis coincides with the direction of the component of the magnetic field at right angles to the Z-axis. The projection of the path on the plane X_1Y_1 has the semiaxes a and b. II. The projection of the path on the plane Y_1Z has the semiaxes b and c.

The complete solution of (1) is given by the sum of (2), (3), (4) and (2a), (3a), (4a), and the solution may be brought to satisfy an arbitrary initial state, partly by means of the three values of C, namely C_1, C_2 and C_3, and partly by means of B_1 (after which (5) determines B_2 and B_3), A_1 (after which (7) determines A_2 and A_3) and φ_1 (after which (8) determines φ_2 and φ_3).

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We shall next determine the path corresponding to the particular solution given by the equations (2), (3) and (4), *i. e.* the forced oscillations alone, and we take the angle φ equal to $-\frac{\pi}{2}\cdot$ We introduce a new system of co-ordinates (see Fig. 1) by rotating the X- and Y-axis an angle φ_0 around the Z-axis into the new position X_1, Y_1, the angle φ_0 being so chosen that

$$\cos\varphi_0 = \frac{h_x}{\sqrt{h_x^2 + h_y^2}} \quad \text{and} \quad \sin\varphi_0 = \frac{h_y}{\sqrt{h_x^2 + h_y^2}}\cdot$$

The magnetic field may then be considered as composed of the two components H_z in the direction of the Z-axis and $H_{x_1} = \sqrt{H_x^2 + H_y^2}$ in the direction of the X_1-axis. In accordance with the above we have

$$h_{x_1} = \sqrt{h_x^2 + h_y^2} \quad \text{and} \quad h = \sqrt{h_{x_1}^2 + h_z^2}.$$

By elimination of t from the equations (2), (3) and (4) and inserting $\varphi = -\frac{\pi}{2}$ we find:

$$\frac{y_1^2}{\left(\frac{h_{x_1}}{\omega} q\right)^2} + \frac{z^2}{\left(\frac{h_z^2 - \omega^2}{\omega^2} q\right)^2} = 1, \quad \text{where } q = \frac{\mu E_0}{\omega^2 - h^2} = \frac{e E_0}{m(\omega^2 - h^2)}, \tag{9}$$

and

$$\frac{x_1}{h_{x_1} \cdot h_z} = \frac{z}{h_z^2 - \omega^2}. \tag{10}$$

The path is thus lying in the plane through the Y_1-axis, determined by (10) (see Fig. 2) and its projection on the Y_1Z-plane is an ellipse with the following semi-axes (see Fig. 1. II):

$$b = \left|\frac{h_{x_1}}{\omega} q\right| \quad \text{and} \quad c = \left|\frac{h_z^2 - \omega^2}{\omega^2} q\right|. \tag{9a}$$

The projection of the path on the X_1Y_1-plane is an ellipse with the semi-axes (see Fig. 1, I):

$$a = \left|\frac{h_{x_1} h_z}{\omega^2} q\right| \quad \text{and} \quad b = \left|\frac{h_{x_1}}{\omega} q\right|, \tag{9b}$$

while the semi-axes of the path itself are

$$b = \left|\frac{h_{x_1}}{\omega} q\right| \quad \text{and} \quad \sqrt{a^2 + c^2} = \frac{\sqrt{h_{x_1}^2 h_z^2 + (h_z^2 - \omega^2)^2}}{\omega^2} \cdot |q|. \tag{9c}$$

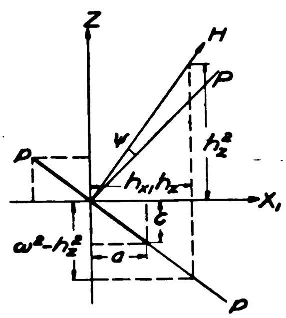

Fig. VII. 2. The Elliptic Path is lying in the plane pp, and the direction H of the magnetic field forms the angle ψ with the perpendicular P to this plane.

The direction H of the magnetic field (see Fig. 2) forms the angle ψ with the perpendicular P to the plane of the path where

$$\operatorname{tg}\psi = \frac{h_z}{h_{x_1}} \cdot \frac{\omega^2 - h^2}{\omega^2}. \tag{10a}$$

From (9c) it appears that both semiaxes approach infinity when ω approaches h. Since the elliptic path is traversed once for each period, and since the area of the ellipse and the velocity of the particle thus approach infinity when $\omega \rightarrow h$, an ion may under these conditions exert a great influence on

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the propagation of the waves, since it highly affects the conductivity and the dielectric constant of the medium and, thereby, also the polarization, refraction and attenuation of the waves. These are the points of view forming the basis of the above mentioned authors' considerations concerning the influence exerted by the earth's magnetic field on the propagation of the waves. However, the method is incomplete since it disregards the free oscillations of the ion in the magnetic field and, consequently, also disregards the time used in working up the forced oscillations, and this time approaches infinity[1] as $\omega \rightarrow h$. We shall therefore examine these conditions more closely in a few simple cases. For this purpose we commence by assuming:

$$H_y = H_z = 0; \quad h_y = h_z = 0; \quad \text{consequently } h_x = h = \frac{e}{mc} H_{x0}.$$

The equations (I') are in this case reduced to

$$\frac{d^2x}{dt^2} = 0, \quad \frac{d^2y}{dt^2} = h\frac{dz}{dt}, \quad \frac{d^2z}{dt^2} = \mu E_0 \cos(\omega t + \varphi) - h\frac{dy}{dt}, \tag{I''}$$

and the forced oscillations become:

$$x = 0, \quad y = -\frac{h}{\omega} q \sin(\omega t + \varphi), \quad z = -q\cos(\omega t + \varphi), \tag{11}$$

where as usual $q = \frac{eE_0}{m(\omega^2 - h^2)}$.

For the free oscillations we have:

$$A_1 = 0, \quad A_2 = A_3 = A; \quad B_1 = B; \quad B_2 = B_3 = 0. \tag{11a}$$

Furthermore we have $\cos(\varphi_2 - \varphi_3) = 0$, which equation is satisfied by

$$\varphi_2 = \varphi' \text{ and by } \varphi_3 = \frac{\pi}{2} + \varphi_2 = \frac{\pi}{2} + \varphi'.$$

The complete solution therefore becomes

$$\left.\begin{aligned} x &= Bt + C_1, \\ y &= A\cos(ht + \varphi') - \frac{h}{\omega} q \sin(\omega t + \varphi) + C_2, \\ z &= -A\sin(ht + \varphi') - q\cos(\omega t + \varphi) + C_3. \end{aligned}\right\} \tag{12}$$

The motion in the direction of the X-axis presents no peculiarities, and will be disregarded in what follows.

We further assume, for $t = 0$:

$$y = 0, \quad z = 0, \quad \frac{dy}{dt} = U_{y0} \text{ and } \frac{dz}{dt} = U_{z0}. \tag{13}$$

According to (12) and (13) we have:

---

[1] *Nichols* and *Schelleng* write (l. c. p. 219): »All time variables are assumed periodic with a frequency $\frac{\omega}{2\pi}$, so that $\frac{\partial}{\partial t} = j\omega$« and (page 220): »It is here assumed that the mean time between collisions is large compared to $\frac{1}{\omega}$«, *i. e.*, with our denominations that $v \ll \omega$.

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$$\left.\begin{aligned} A\cos\varphi' - \frac{h}{\omega} q \sin\varphi + C_2 &= 0; \\ -A\sin\varphi' - q\cos\varphi + C_3 &= 0; \\ -hA\sin\varphi' - hq\cos\varphi &= U_{y0}; \\ -hA\cos\varphi' + \omega q\sin\varphi &= U_{z0}, \end{aligned}\right\} \quad (14$$

whence:

$$\left.\begin{aligned} &C_3 = -\frac{U_{y0}}{h}, \quad C_2 = \frac{U_{z0}}{h} - \frac{\omega^2 - h^2}{\omega h} q\sin\varphi, \\ &A\sin\varphi' = -\frac{U_{y0}}{h} - q\cos\varphi, \quad A\cos\varphi' = -\frac{U_{z0}}{h} + \frac{\omega}{h} q\sin\varphi. \end{aligned}\right\} \quad (14a$$

Inserting these values into (12) we find the general solution for the motion of the ion corresponding to the initial state determined by (13). The equations of motion therefore are:

$$\left.\begin{aligned} y &= \left(-\frac{U_{z0}}{h} + \frac{\omega}{h} q\sin\varphi\right)\cos(ht) + \left(\frac{U_{y0}}{h} + q\cos\varphi\right)\sin(ht) \\ &\quad - \frac{h}{\omega} q\sin(\omega t + \varphi) + \frac{U_{z0}}{h} - \frac{\omega^2 - h^2}{\omega h} q\sin\varphi, \\ z &= \left(\frac{U_{y0}}{h} + q\cos\varphi\right)\cos(ht) + \left(\frac{U_{z0}}{h} - \frac{\omega}{h} q\sin\varphi\right)\sin(ht) \\ &\quad - q\cos(\omega t + \varphi) - \frac{U_{y0}}{h}, \end{aligned}\right\} \quad (15.$$

and correspondingly:

$$\left.\begin{aligned} \frac{dy}{dt} &= (U_{z0} - \omega q\sin\varphi)\sin(ht) + (U_{y0} + hq\cos\varphi)\cos(ht) \\ &\quad - hq\cos(\omega t + \varphi), \\ \frac{dz}{dt} &= -(U_{y0} + hq\cos\varphi)\sin(ht) + (U_{z0} - \omega q\sin\varphi)\cos(ht) \\ &\quad + \omega q\sin(\omega t + \varphi). \end{aligned}\right\} \quad (16$$

For $\omega = h$ equations (15) and (16) assume the indeterminate form $\frac{0}{0}$ which may be reduced to

$$\left.\begin{aligned} y &= \frac{U_{z0}}{h}(1 - \cos(ht)) + \frac{U_{y0}}{h}\sin(ht) + \frac{\mu E_0}{h^2}(\sin\varphi\cos(ht) + \tfrac{1}{2}\cos\varphi\sin(ht) \\ &\quad - \sin\varphi) - \frac{t}{2h}\mu E_0\cos(ht + \varphi), \\ z &= -\frac{U_{y0}}{h}(1 - \cos(ht)) + \frac{U_{z0}}{h}\sin(ht) - \frac{\mu E_0}{2h^2}\sin\varphi\sin(ht) \\ &\quad + \frac{t\mu E_0}{2h}\sin(ht + \varphi), \end{aligned}\right\} \quad (15a)$$

and

$$\left.\begin{aligned} \frac{dy}{dt} &= U_{y0}\cos(ht) + U_{z0}\sin(ht) - \frac{\mu E_0}{2h}\sin\varphi\sin(ht) + \frac{t\mu E_0}{2}\sin(ht + \varphi), \\ \frac{dz}{dt} &= U_{z0}\cos(ht) - U_{y0}\sin(ht) + \frac{\mu E_0}{2h}\cos\varphi\sin(ht) + \frac{t\mu E_0}{2}\cos(ht + \varphi). \end{aligned}\right\} \quad (16a)$$

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For very high values of t (15a) and (16a) approach the following limiting values:

$$\left.\begin{aligned} y &= -\frac{t\mu E_0}{2h}\cos(ht+\varphi), \\ z &= \frac{t\mu E_0}{2h}\sin(ht+\varphi), \end{aligned}\right\} \tag{15b}$$

and

$$\left.\begin{aligned} \frac{dy}{dt} &= \frac{t\mu E_0}{2}\sin(ht+\varphi), \\ \frac{dz}{dt} &= \frac{t\mu E_0}{2}\cos(ht+\varphi), \end{aligned}\right\} \tag{16b}$$

showing that the path approaches a circle of radius

$$r_t = \frac{t\mu E_0}{2h}, \tag{17}$$

and the circular orbit is traversed with the velocity

$$U_t = \frac{t\mu E_0}{2}. \tag{18}$$

It is easy to see that the direction of motion of the ion is such that the ion, by its motion, counteracts the magnetic field in the interior of the orbit, and the Z-component of the velocity of the ion is in phase with the electric force. In the present case where $\omega = h$ and where simultaneously $ht \gg 1$, the forced oscillations will thus be characteristic for the motion, and the path and velocity of the ion will be relatively very great.

The conditions will be quite different if the time between two collisions is so short that $ht \ll 1$ as well as $\omega t \ll 1$. In that case, whether ω be equal to h or $\omega \gtrless h$, we find in either case, *i. e.* both for (15a) and for (15):

$$\left.\begin{aligned} y &= tU_{y0} + \tfrac{1}{2}ht^2U_{z0}, \\ z &= tU_{z0} - \tfrac{1}{2}ht^2U_{y0} + \tfrac{1}{2}t^2\mu E_0\cos\varphi. \end{aligned}\right\} \tag{15c}$$

Similarly, both for $\omega = h$ and for $\omega \gtrless h$, the velocities immediately after a collision will be determined by

$$\left.\begin{aligned} \frac{dy}{dt} &= U_{y0} + htU_{z0}, \\ \frac{dz}{dt} &= U_{z0} - htU_{y0} + t\mu E_0\cos\varphi. \end{aligned}\right\} \tag{16c}$$

If the time between two consecutive collisions is so short that $\omega t \ll 1$ as well as $ht \ll 1$, the motion of the ion will tend to be independent of the frequency of the electric field, as shown by the equations (15c) and (16c).

If in accordance with *Nichols* and *Schelleng*[1] we assume waves of wavelength 214 m to be in »resonance« with an electron travelling in the earth's magnetic field, then the corresponding period will be

$$\frac{2\pi}{h} = \frac{2\pi}{\omega} = \frac{214}{3\cdot 10^8} = 7.1\cdot 10^{-7}\ \text{sec, or } h = \text{abt. } 9\cdot 10^6.$$

[1] l. c., p. 218.

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The mean number η of periods during the time between two collisions is consequently

$$\eta = \frac{\text{Time between two Collisions}}{\text{Duration of Period}} = \frac{1}{v} \cdot \frac{1}{7.1 \cdot 10^{-7}}$$

where v, as previously, indicates the average number of collisions per second suffered by the ion. For the atmospheres F and F', see Chapter IV, table 3, we find the following values of η respectively η':

Height =	60	80	100	120 km
η_{el} =	0.035	0.71	10.6	46
η'_{el} =	0.035	0.73	14.6	200

These values of η and η' show that not until a height of about 100 km or more has been reached will the time between two collisions be many times greater than the period. Below 60 or 80 km the motions of the electrons will therefore not be governed by the formulas (15b) and (16b), and will consequently not to any great extent be influenced by the forced oscillations, but the motions will be represented much more accurately by the formulas (15c) and (16c) which are independent of the frequency of the waves.

On the other hand the above values of η show that for heights above 70 or 80 km it is not justifiable to assume $ht \ll 1$. We shall therefore make a general investigation of the influence of the magnetic field on some of the factors governing the propagation of radio waves while also taking into account the collisions of the ions with the neutral molecules.

2. *The Influence of Ions and Electrons on the Conductivity and the Dielectric Constant when the Propagation of the Wave is Parallel to the Magnetic Field.*

By the motion of the waves in the direction of the magnetic field a rotation of the plane of polarization of the waves will be brought about. In determining this rotation we assume in the usual manner that the linearly polarized wave, with an electric field in the YZ-plane determined by

$$E_z = E_0 \cos(\omega t + \varphi), \tag{19}$$

is composed of two circularly polarized oscillations E_I and E_{II} the components of which in the direction of the Y-axis are, respectively

$$E_y = E_0 \sin(\omega t + \varphi) \quad \text{and} \quad -E_y = -E_0 \sin(\omega t + \varphi). \tag{20}$$

We write therefore

$$E_I = E_z + E_y \quad \text{and} \quad E_{II} = E_z - E_y, \tag{21}$$

and consequently we have that

$$E_z = E_0 \cos(\omega t + \varphi) = \tfrac{1}{2}(E_I + E_{II}). \tag{22}$$

The electric field in the two circular oscillations E_I and E_{II} corresponds to a rotation, with a constant angular velocity ω, of the electric-force vector E_0 in the directions of rotation indicated by the arrows (Fig. 3).

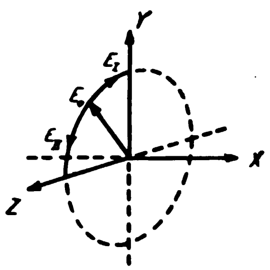

Fig. VII. 3. The Electric Field in the two Circularly Polarized Oscillations $E_I = E_z + E_y$ and $E_{II} = E_z - E_y$, where $E_z = E_0 \cos(\omega t + \varphi)$ and $E_y = E_0 \sin(\omega t + \varphi)$ may be imagined to be produced by the electric-force vector E_0 rotating at angular velocity ω in the two directions of rotation indicated by the arrows.

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We shall first consider the propagation of the plane circularly polarized wave corresponding to E_I.

In this case the equation (I′) becomes:

$$\left.\begin{aligned}\frac{d^2y}{dt^2} &= \mu E_0 \sin(\omega t + \varphi) + h\frac{dz}{dt},\\ \frac{d^2z}{dt^2} &= \mu E_0 \cos(\omega t + \varphi) - h\frac{dy}{dt}.\end{aligned}\right\}\qquad (I'')$$

It is easy to show that these equations are satisfied by

$$\left.\begin{aligned}y &= \left[-\frac{U_{z0}}{h} + q\frac{h+\omega}{h}\sin\varphi\right]\cos(ht) + \left[\frac{U_{y0}}{h} + q\frac{h+\omega}{h}\cos\varphi\right]\sin(ht)\\ &\quad - q\frac{h+\omega}{\omega}\sin(\omega t + \varphi) + \frac{U_{z0}}{h} - \frac{\omega^2 - h^2}{\omega h} q\sin\varphi,\\ &\text{and}\\ z &= \left[\frac{U_{y0}}{h} + q\frac{h+\omega}{h}\cos\varphi\right]\cos(ht) + \left[\frac{U_{z0}}{h} - q\frac{h+\omega}{h}\sin\varphi\right]\sin(ht)\\ &\quad - q\frac{h+\omega}{\omega}\cos(\omega t + \varphi) - \frac{U_{y0}}{h} - \frac{\omega^2 - h^2}{\omega h} q\cos\varphi,\end{aligned}\right\}\qquad (23)$$

corresponding to:

$$\left.\begin{aligned}\frac{dy}{dt} &= [U_{z0} - q(h+\omega)\sin\varphi]\sin(ht) + [U_{y0} + q(h+\omega)\cos\varphi]\cos(ht)\\ &\quad - q(h+\omega)\cos(\omega t + \varphi),\\ &\text{and}\\ \frac{dz}{dt} &= -[U_{y0} + q(h+\omega)\cos\varphi]\sin(ht) + [U_{z0} - q(h+\omega)\sin\varphi]\cos(ht)\\ &\quad + q(h+\omega)\sin(\omega t + \varphi),\end{aligned}\right\}\qquad (24)$$

and this solution gives for $t = 0$: $y = z = 0$; $\frac{dy}{dt} = U_{y0}$ and $\frac{dz}{dt} = U_{z0}$.

From (24) follows:

$$\left.\begin{aligned}\left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2 &= [U_{y0} + q(h+\omega)\cos\varphi]^2 + [U_{z0} - q(h+\omega)\sin\varphi]^2 + q^2(h+\omega)^2\\ &\quad + 2q(h+\omega)\Big\{[U_{z0} - q(h+\omega)\sin\varphi]\sin((\omega - h)t + \varphi)\\ &\quad - [U_{y0} + q(h+\omega)\cos\varphi]\cos((\omega - h)t + \varphi)\Big\}.\end{aligned}\right\}\qquad (25)$$

Taking the mean value of this, while taking into consideration the equal probability of positive and negative values of U_{y0} and U_{z0} and also that all values of φ between 0 and 2π are similarly probable, we find:

$$\overline{\left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} = U_{y0}^2 + U_{z0}^2 + 2q^2(\omega + h)^2[1 - \cos(\omega - h)t]. \qquad (26)$$

Proceeding then in the same manner as in Chapter VI, equations (15)—(26), we derive without difficulty:

$$\sigma_I = \frac{e^2}{m}\cdot\frac{\nu}{\nu^2 + (\omega - h)^2}\ [\text{e. s. u.}] = \frac{e^2}{m c^2}\cdot\frac{\nu}{\nu^2 + (\omega - h)^2}\ [\text{e. m. u.}]. \qquad (27)$$

The corresponding value for the circularly polarized wave II may evidently be obtained simply by substituting $-h$ for h. Consequently we have:

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$$\sigma_{II} = \frac{e^2}{m} \cdot \frac{\nu}{\nu^2 + (\omega + h)^2} \quad [\text{e. s. u.}] = \frac{e^2}{mc^2} \cdot \frac{\nu}{\nu^2 + (\omega + h)^2} \quad [\text{e. m. u.}]. \tag{28}$$

We shall next, in accordance with Chapter VI, find the mean amplitude $\overline{\Phi_z(t)}$ of the component of the velocity $\left(\frac{dz}{dt}\right)$ which is in phase with $\sin(\omega t + \varphi)$ and consequently an angle $\frac{\pi}{2}$ behind the acting force $E_z = E_0 \cos(\omega t + \varphi)$. For this purpose we write preliminarily

$$\left.\begin{aligned}
\Psi_z(t) &= 2\int_0^t \frac{dz}{dt} \sin(\omega t + \varphi)\, dt \\
&= [U_{y0} + q(h+\omega)\cos\varphi]\left(\frac{\sin((\omega+h)t+\varphi)}{\omega+h} - \frac{\sin((h-\omega)t-\varphi)}{h-\omega} - \frac{\sin\varphi}{h+\omega} - \frac{\sin\varphi}{h-\omega}\right) \\
&- [U_{z0} - q(h+\omega)\sin\varphi]\left(\frac{\cos((\omega+h)t+\varphi)}{\omega+h} + \frac{\cos((\omega-h)t+\varphi)}{\omega-h} - \frac{\cos\varphi}{h+\omega} + \frac{\cos\varphi}{h-\omega}\right) \\
&+ q(h+\omega)\left[t - \frac{1}{2\omega}\sin 2(\omega t + \varphi) + \frac{1}{2\omega}\sin 2\varphi\right].
\end{aligned}\right\} \tag{29}$$

The mean value of this, as φ passes through all values from $\varphi = 0$ to $\varphi = 2\pi$, is

$$\Phi_z(t) = \frac{1}{2\pi}\int_0^{2\pi} \Psi_z(t)\, d\varphi = q(\omega + h)\left(t - \frac{\sin(\omega - h)t}{\omega - h}\right). \tag{30}$$

It is easy to show that the same value is obtained for $\Phi_y(t)$.

The mean amplitude of that velocity component which in phase is an angle $\frac{\pi}{2}$ behind the acting force is thus, for the free paths traversed during the time t, determined by:

$$\overline{\Phi_I(t)} = \frac{1}{t} \cdot \Phi(t) = q(\omega + h)\left(1 - \frac{\sin(\omega - h)t}{(\omega - h)t}\right) \tag{31}$$

for the Z-direction as well as for the Y-direction.

By proceeding in the same manner as in Chapter VI, equations (30) to (36), we easily find:

$$\Delta_I \varepsilon = 4\pi \frac{e^2}{m} \cdot \frac{\omega - h}{\omega\left(\nu^2 + (\omega - h)^2\right)}, \tag{32}$$

and correspondingly for wave II

$$\Delta_{II} \varepsilon = 4\pi \frac{e^2}{m} \cdot \frac{\omega + h}{\omega\left(\nu^2 + (\omega + h)^2\right)}. \tag{33}$$

For $h^2 = \omega^2 - \nu^2$ or $\nu^2 = \omega^2 - h^2$ we have

$$\Delta_I \varepsilon = \Delta_{II} \varepsilon = 4\pi \frac{e^2}{m} \frac{1}{2\omega^2}. \tag{34}$$

If there are N ions or electrons per cm³, the equations (27), (28), (32) and (33) show that

$$\sigma_I = N \frac{e^2}{mc^2} \cdot \frac{\nu}{\nu^2 + (\omega - h)^2}, \qquad [\text{e. m. u.}] \tag{35}$$

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$$\sigma_{II} = N \frac{e^2}{mc^2} \cdot \frac{\nu}{\nu^2 + (\omega + h)^2}, \qquad \text{[e. m. u.]} \qquad (36)$$

$$\varepsilon_I = 1 - \Delta_I \varepsilon = 1 - N 4\pi \frac{e^2}{m} \cdot \frac{\omega - h}{\omega(\nu^2 + (\omega - h)^2)} \qquad \text{[e. s. u.]} \qquad (37)$$

and

$$\varepsilon_{II} = 1 - \Delta_{II} \varepsilon = 1 - N 4\pi \frac{e^2}{m} \cdot \frac{\omega + h}{\omega(\nu^2 + (\omega + h)^2)}. \qquad \text{[e. s. u.]} \qquad (38)$$

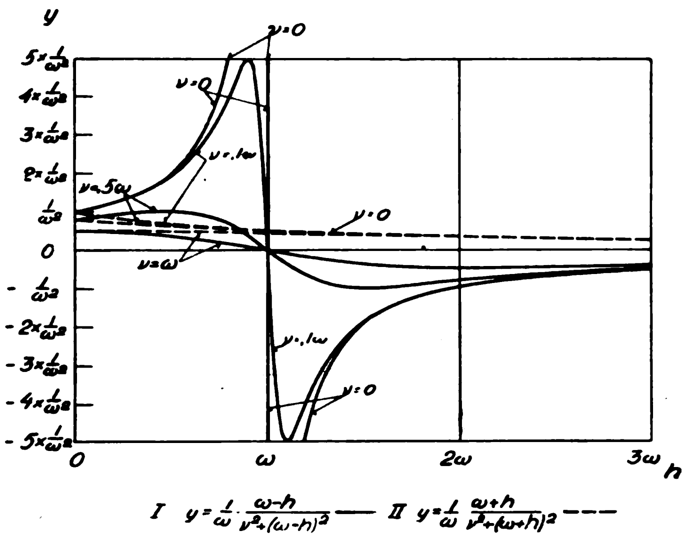

Fig. VII. 4. The Value of $\frac{m\Delta_I\varepsilon}{4\pi e^2}$ and of $\frac{m\Delta_{II}\varepsilon}{4\pi e^2}$ as a Function of *h*.

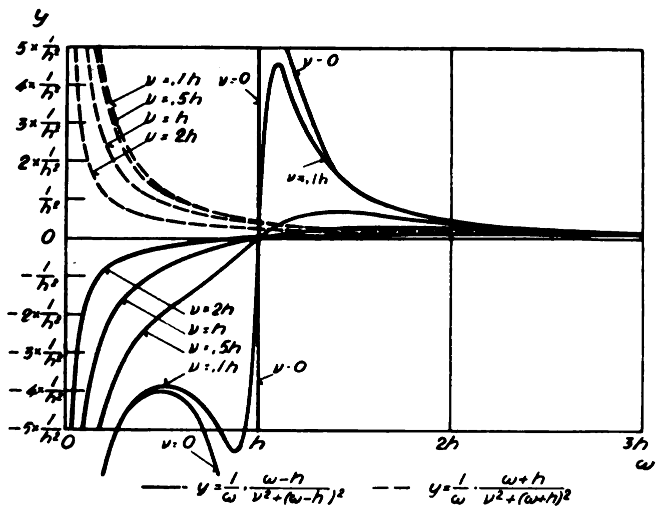

Fig. VII. 5. The Value of $\frac{m\Delta_I\varepsilon}{4\pi e^2}$ and of $\frac{m\Delta_{II}\varepsilon}{4\pi e^2}$ as a Function of ω.

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In further consequence of this we have

$$\frac{\sigma_{\mathrm{I}}}{\Delta_{\mathrm{I}}\varepsilon} = \frac{1}{4\pi c^2} \cdot \frac{v\omega}{\omega - h} \quad \text{and} \quad \frac{\sigma_{\mathrm{II}}}{\Delta_{\mathrm{II}}\varepsilon} = \frac{1}{4\pi c^2} \cdot \frac{v\omega}{\omega + h}. \tag{39}$$

For $h = 0$ these equations are reduced to the corresponding equations (42) to (45) in Chapter VI.

We have in these calculations assumed the propagation of the waves to follow the positive direction of the magnetic field. If we had chosen the opposite direction of propagation, we should have arrived at the same formulas, only with h replaced by $-h$. This corresponds to the cases I and II being interchanged. We may therefore, without limiting the problem in any way, always consider h to be positive in the formulas above.

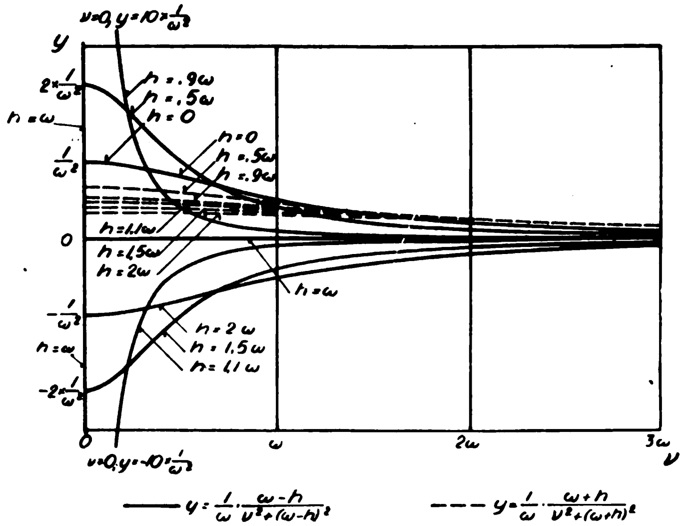

Fig. VII. 6. The Value of $\frac{m\Delta_{\mathrm{I}}\varepsilon}{4\pi e^2}$ and of $\frac{m\Delta_{\mathrm{II}}\varepsilon}{4\pi e^2}$ as a Function of v.

For $v = 0$ the equations (37) and (38) will be reduced to

$$\Delta_{\mathrm{I}}\varepsilon = N 4\pi \frac{e^2}{m} \frac{1}{\omega(\omega - h)} \quad \text{and} \quad \Delta_{\mathrm{II}}\varepsilon = N 4\pi \frac{e^2}{m} \frac{1}{\omega(\omega + h)} \tag{40) (41}$$

agreeing with the values found by *Appleton* and *Barnett*[1], by *Nichols* and *Schelleng*[2] and by *A. Hoyt Taylor* and *E. O. Hulburt*[3], and used also by *G. Breit* and *M. A. Tuve*[4] in their investigations, to which we shall come back in the following.

Figs. 4 to 6 show $\Delta_{\mathrm{I}}\varepsilon$ and $\Delta_{\mathrm{II}}\varepsilon$ as functions of h, ω and v, respectively. An inspection of these Figures shows that the approximation formulas for $\Delta_{\mathrm{I}}\varepsilon$

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[1] l. c. equation (2), p. 674.

[2] l. c. equations (7) and (8), p. 224.

[3] *A. Hoyt Taylor* and *E. O. Hulburt:* Phys. Rev. (II). Vol. 27, p. 189—215. 1926. (Equations (2) and (3)).

[4] *G. Breit* and *M. A. Tuve:* Phys. Rev. (II). Vol. 28, p. 554—575. 1926. (Equations (5′) and (6′)).

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given by (40) offer a most imperfect approximation even for rather low values of v.

A comparison between (39) and equation (42) in Chapter VI shows that for wave II $\left(\frac{\sigma}{\Delta\varepsilon}\right)$ is always smaller with a magnetic field than without this field. For wave I and for $h < 2\omega$, on the contrary, $\left|\frac{\sigma}{\Delta\varepsilon}\right|$ is always larger with than without a magnetic field, and if h does not differ much from ω, the magnetic field will cause a large increase in this ratio. It should further be noted that for $h > \omega$ $\Delta_{\mathrm{I}}\varepsilon$ is negative and, consequently, $\varepsilon_{\mathrm{I}} > 1$.

3. *The Influence of Ions and Electrons on the Conductivity and the Dielectric Constant when the Propagation of the Waves is Perpendicular to the Magnetic Field.*

We assume, as above, that the waves move in the direction of the X-axis, and that their electric field in the element of space considered is given by $E_x = E_y = 0$ and $E_z = E_0 \cos(\omega t + \varphi)$. We shall consider the two cases where the magnetic field is either parallel to the electric vector (α) or perpendicular thereto (β).

In case (α) we have: $H_x = H_y = 0$; $H_z = H_0$; $h_z = h = \frac{e}{mc} H_0$; and in case ($\beta$): $H_x = H_z = 0$; $H_y = H_0$; $h_y = h = \frac{e}{mc} H_0$.

Case (α). The differential equations for the motion of the particle are

$$\frac{d^2x}{dt^2} = h\frac{dy}{dt}, \quad \frac{d^2y}{dt^2} = -h\frac{dx}{dt}, \quad \frac{d^2z}{dt^2} = \mu E_0 \cos(\omega t + \varphi). \tag{I''''}$$

From this it is evident that the motion in the direction of the Z-axis goes on exactly as if the magnetic field did not exist. The projection on the XY-plane of the path of the particle consists of circular arcs altered in direction by each collision, but entirely independent of the action of the wave. The propagation of the wave will therefore take place in the same manner as without the field. The influence of the ions and electrons on the conductivity and dielectric constant therefore remains the same as without the magnetic field, so that in this case it will be sufficient to refer to Chapter VI.

Case (β). The differential equations for the motion of the particle are:

$$\frac{d^2x}{dt^2} = -h\frac{dz}{dt}, \quad \frac{d^2y}{dt^2} = 0, \quad \frac{d^2z}{dt^2} = \mu E_0 \cos(\omega t + \varphi) + h\frac{dx}{dt}. \tag{I$^{\text{v}}$}$$

These equations are satisfied by:

$$\left.\begin{aligned} x &= \left[\frac{U_{z0}}{h} - \frac{\omega}{h} q \sin\varphi\right]\cos(ht) + \left[\frac{U_{x0}}{h} - q\cos\varphi\right]\sin(ht) + \frac{h}{\omega} q \sin(\omega t + \varphi) \\ &\qquad - \frac{U_{z0}}{h} + \frac{\omega^2 - h^2}{\omega h} q \sin\varphi, \\ z &= -\left[\frac{U_{x0}}{h} - q\cos\varphi\right]\cos(ht) + \left[\frac{U_{z0}}{h} - \frac{\omega}{h} q\sin\varphi\right]\sin(ht) - q\cos(\omega t + \varphi) + \frac{U_{x0}}{h} \end{aligned}\right\} \tag{42}$$

and correspondingly:

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$$\left.\begin{aligned}\frac{dx}{dt} &= -[U_{z0} - \omega q \sin\varphi]\sin(ht) + [U_{x0} - hq\cos\varphi]\cos(ht) + hq\cos(\omega t + \varphi),\\ \frac{dz}{dt} &= [U_{x0} - hq\cos\varphi]\sin(ht) + [U_{z0} - \omega q\sin\varphi]\cos(ht) + \omega q\sin(\omega t + \varphi),\end{aligned}\right\} \quad (43)$$

since for $t = 0$ we have: $x = z = 0$ and $\frac{dx}{dt} = U_{x0}$, $\frac{dz}{dt} = U_{z0}$.

We may now proceed exactly in the same manner as above, and find then that the increase in the conductivity caused by one ion per cm³ is determined by

$$\sigma_{\perp} = \frac{e^2}{mc^2} \cdot \frac{\nu(\nu^2 + \omega^2 + h^2)}{(\nu^2 + \omega^2 + h^2)^2 - 4\omega^2 h^2} = \tfrac{1}{2}(\sigma_{\mathrm{I}} + \sigma_{\mathrm{II}}). \quad \text{[e. m. u.]} \quad (44)$$

. In exactly similar manner the decrease in the dielectric constant caused by one ion per cm³ is determined to be

$$\Delta_{\perp}\varepsilon = 4\pi\frac{e^2}{m} \cdot \frac{\nu^2 + \omega^2 - h^2}{(\nu^2 + \omega^2 + h^2)^2 - 4\omega^2 h^2} = \tfrac{1}{2}(\Delta_{\mathrm{I}}\varepsilon + \Delta_{\mathrm{II}}\varepsilon), \quad \text{[e. s. u.]} \quad (45)$$

which differs from the values of $\Delta_{\mathrm{I}}\varepsilon$ and $\Delta_{\mathrm{II}}\varepsilon$ determined by the equations (32) and (33), and applying to the circularly polarized waves in the case where the direction of propagation of the waves is parallel to the direction of the magnetic field.

For the case $\nu = 0$ equation (45) is reduced to:

$$\Delta_{\perp}\varepsilon = 4\pi\frac{e^2}{m}\,\frac{1}{\omega^2 - h^2}. \quad (45a)$$

In the formulas of *Appleton* and *Barnett*[1], *Taylor* and *Hulburt*[2] as well as *Breit* and *Tuve*[3] corresponding to (45a) there is added another term in the denominator corresponding to the polarization[4] occurring in case of solid bodies, but no such polarization takes place in the present case.

If there are N ions per unit volume (45) gives:

$$\Delta_{\perp}\varepsilon = N4\pi\frac{e^2}{m} \cdot \frac{\nu^2 + \omega^2 - h^2}{(\nu^2 + \omega^2 + h^2)^2 - 4\omega^2 h^2}. \quad (45b)$$

Here too the approximation obtained by inserting in (45) and (45b) $\nu = 0$ is of doubtful value. According to (45a) we find:

$$\Delta_{\perp}\varepsilon_{(\nu=0)} = N4\pi\frac{e^2}{m} \cdot \frac{1}{\omega^2 - h^2}. \quad (45c)$$

If on the other hand we insert in (45b) $\omega = h$, we find:

$$\Delta_{\perp}\varepsilon_{(\omega=h)} = N4\pi\frac{e^2}{m} \cdot \frac{1}{\nu^2 + 4h^2}, \quad (45d)$$

which for $\nu \to 0$ is reduced to

$$\Delta_{\perp}\varepsilon_{\left(\substack{\omega=h\\ \nu\to 0}\right)} = N4\pi\frac{e^2}{m}\,\frac{1}{4h^2} \quad (45e)$$

which is the true value of $\Delta_{\perp}\varepsilon$ for $\omega = h$ and for very small values of ν while (45c) for $\omega = h$ leads to $\Delta_{\perp}\varepsilon_{(\nu=0)} = \pm\infty$.

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[1] l. c., equation (4). [2] l. c., equation (6). [3] l. c., equation (8').
[4] *H. A. Lorentz:* The Theory of Electrons, Chapter IV. (1916).

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In the case considered here, where the direction of the magnetic field is perpendicular to the direction of propagation of the waves, the velocity of propagation of the waves will depend on their polarization, since this velocity in general will be different for waves with the electric vector parallel to the magnetic field and at right angle thereto (double refraction).

When $\omega \to 0$ we get

$$\sigma_{I} = \sigma_{II} = \sigma_{\perp} = N \cdot \frac{e^2}{mc^2} \cdot \frac{\nu}{\nu^2 + h^2} \quad \text{and} \quad \Delta_{\perp}\varepsilon = N \cdot 4\pi \frac{e^2}{m} \cdot \frac{\nu^2 - h^2}{(\nu^2 + h^2)^2}. \tag{45f}$$

4. *Rotation of the Plane of Polarization.*

The plane circularly polarized waves I and II will move in the direction of the X-axis at phase velocities v_I and v_{II} determined by

$$v_I = \frac{c}{n_I} \quad \text{and} \quad v_{II} = \frac{c}{n_{II}}, \tag{46}$$

where n_I and n_{II}, as will be shown in Chapter VIII, are determined by:

$$n_I = \sqrt{\frac{\varepsilon_I}{2} + \sqrt{\frac{\varepsilon_I^2}{4} + \left(2\pi c^2 \frac{\sigma_I}{\omega}\right)^2}} \quad \text{and} \quad n_{II} = \sqrt{\frac{\varepsilon_{II}}{2} + \sqrt{\frac{\varepsilon_{II}^2}{4} + \left(2\pi c^2 \frac{\sigma_{II}}{\omega}\right)^2}}. \tag{47}$$

Inserting here the values of ε_I, ε_{II}, σ_I and σ_{II} from the formulas (35) to (38) we find:

$$n_I^2 = \frac{1}{2} - \frac{k(\omega - h)}{\omega(\nu^2 + (\omega - h)^2)} + \sqrt{\frac{1}{4} + \frac{k}{\omega^2} \cdot \frac{k - \omega(\omega - h)}{\nu^2 + (\omega - h)^2}} \tag{48}$$

and

$$n_{II}^2 = \frac{1}{2} - \frac{k(\omega + h)}{\omega(\nu^2 + (\omega + h)^2)} + \sqrt{\frac{1}{4} + \frac{k}{\omega^2} \cdot \frac{k - \omega(\omega + h)}{\nu^2 + (\omega + h)^2}} \tag{49}$$

where

$$k = N 2\pi \frac{e^2}{m} \tag{50}$$

and N is the number of ions per unit volume.

In the plane determined by $x = l$ the Z-component of the electric force in wave I is consequently $E_0 \cos\left(\omega t + \varphi - \omega n_I \frac{l}{c}\right)$, and the Y-component is $E_0 \sin\left(\omega t + \varphi - \omega n_I \frac{l}{c}\right)$, while for wave II the Z-component is $E_0 \cos\left(\omega t + \varphi - \omega n_{II} \frac{l}{c}\right)$ and the Y-component $- E_0 \sin\left(\omega t + \varphi - \omega n_{II} \frac{l}{c}\right)$. The Z-component of the resultant oscillation is consequently:

$$\left.\begin{aligned} &\tfrac{1}{2} E_0 \left(\cos\left(\omega t + \varphi - \omega n_I \frac{l}{c}\right) + \cos\left(\omega t + \varphi - \omega n_{II} \frac{l}{c}\right)\right) \\ &= E_0 \cos\left(\frac{\omega l}{2c}(n_{II} - n_I)\right) \cdot \cos\left(\omega t + \varphi - (n_I + n_{II})\frac{\omega l}{2c}\right), \end{aligned}\right\} \tag{51}$$

and the Y-component is

$$\left.\begin{aligned} &\tfrac{1}{2} E_0 \left(\sin\left(\omega t + \varphi - \omega n_I \frac{l}{c}\right) - \sin\left(\omega t + \varphi - \omega n_{II} \frac{l}{c}\right)\right) \\ &= E_0 \sin\left(\frac{\omega l}{2c}(n_{II} - n_I)\right) \cdot \cos\left(\omega t + \varphi - (n_I + n_{II})\frac{\omega l}{2c}\right). \end{aligned}\right\} \tag{52}$$

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A comparison between (51) and (52) shows the resultant wave to be constantly linearly polarized, but the plane of polarization has rotated through the angle θ determined by

$$\theta = \frac{\omega l}{2c}(n_{II} - n_I) \tag{53}$$

and the rotation θ_0 per unit length is consequently

$$\theta_0 = \frac{\theta}{l} = \frac{\omega}{2c}(n_{II} - n_I). \quad [\text{radians cm}^{-1}]. \tag{54}$$

The distance $l_{2\pi}$ travelled by the wave before the plane of polarization is rotated one complete turn is determined by

$$l_{2\pi} = \frac{4\pi c}{\omega} \cdot \frac{1}{|n_{II} - n_I|}. \quad [\text{cm}]. \tag{54a}$$

The formulas (51) to (54) have been deduced on the basis of the assumption that the waves are not attenuated, or at any rate that the two waves are attenuated equally. But they furnish a measure of the rotation of the plane of polarization sufficiently accurate for our purpose.

Inserting $\nu = 0$ into (48) and (49) we find:

$$n_I^2 = 1 - \frac{2k}{\omega(\omega - h)} = 1 - N4\pi\frac{e^2}{m} \cdot \frac{1}{\omega(\omega - h)} \tag{55}$$

and

$$n_{II}^2 = 1 - \frac{2k}{\omega(\omega + h)} = 1 - N4\pi\frac{e^2}{m}\frac{1}{\omega(\omega + h)}. \tag{56}$$

Making the further assumption that both n_I and n_{II} differ only slightly from unity, we easily find that

$$l_{2\pi} = \frac{mc}{Ne^2h}|\omega^2 - h^2|. \qquad [\text{cm}]. \tag{57}$$

For very long waves this results in

$$l_{2\pi(\omega \to 0)} = \frac{mc}{Ne^2}h \qquad [\text{cm}]. \tag{57a}$$

and for very short waves:

$$l_{2\pi(\omega \to \infty)} = \frac{mc}{Ne^2}\frac{\omega^2}{h}. \qquad [\text{cm}]. \tag{57b}$$

Thus for very long waves $l_{2\pi}$ in this case approaches a constant limiting value, while for very short waves it increases with the square of the frequency.

The equations (55) and (56) correspond exactly to those found and used by *Appleton* and *Barnett*[1], *Nichols* and *Schelleng*[2], *Taylor* and *Hulburt*[3] as well as *Breit* and *Tuve*[4]. The magnitude of $l_{2\pi}$ has presumably only been determined by *Nichols* and *Schelleng*[5], and the value found by these authors is about half the value determined according to (57).

The equations (55) to (57b) have been deduced under the assumption that (1) $\nu = 0$ and (2) both n_I and n_{II} differ only slightly from unity. The first assumption corresponds only very imperfectly to actual conditions, and the assumption (2) is

[1] l. c., equation (2), pag. 674.

[2] l. c., equations (7) and (8), p. 224.

[3] l. c., equations (2) and (3).

[4] l. c., equations (5') and (6').

[5] l. c., equation (21).

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very frequently far from being satisfied. These formulas therefore form no reliable basis for estimating the influence of the magnetic field on the rotation of the plane of polarization.

Starting from the accurate formulas (48) and (49) we shall for the sake of orientation deduce some limiting values of $l_{2\pi}$.

We assume $\omega \to 0$, and find then without difficulty

$$l_{2\pi\,(\omega \to 0)} = \frac{4\pi c}{\sqrt{\omega k}} \cdot \frac{\sqrt[4]{v^2 + h^2}}{\sqrt{1 + \frac{h}{\sqrt{v^2 + h^2}}} - \sqrt{1 - \frac{h}{\sqrt{v^2 + h^2}}}} \quad [\text{cm}]. \tag{58}$$

Contrary to the result found in (57a) $l_{2\pi}$ thus increases infinitely, when ω decreases towards zero and this fact will not be altered even if we also assume $v \to 0$; we find then:

$$l_{2\pi\left(\substack{\omega \to 0 \\ v \to 0}\right)} = \frac{4\pi c}{\sqrt{2\omega k}} \sqrt{h}. \quad [\text{cm}] \tag{58a}$$

We shall next consider the »critical« case where $\omega = h$. Assuming in addition $v \to 0$, we find:

$$l_{2\pi\left(\substack{\omega = h \\ v \to 0}\right)} = 4\pi c \sqrt{\frac{v}{hk}}, \quad [\text{cm}] \tag{59}$$

which for $v = 0$ is in agreement with (57).

It is also easily seen that the value of $l_{2\pi}$ determined by formula (57b) for $\omega \to \infty$ is valid in general.

We shall later, in Chapt. IX, present some examples showing how the values of σ_I, σ_{II}, $\sigma_\perp$, $\Delta_I\varepsilon$, $\Delta_{II}\varepsilon$ and $\Delta_\perp\varepsilon$ as well as n_I, n_{II} and $n_\perp$ depend on the height above the earth's surface.

Besides this we shall also consider, in Chapts. VIII, X and XI, the influence of these constants on the attenuation of the waves and the curvature of the paths.

5. *The Values of σ_I, σ_{II}, $\sigma_\perp$, $\Delta_I\varepsilon$, $\Delta_{II}\varepsilon$, and $\Delta_\perp\varepsilon$ for 1 Electron per cubic cm in Case of the Atmosphere F.*

As the formulas previously given for conductivities and changes in dielectric constants caused by electrons and ions are not quite simple when the influence of a magnetic field is taken into consideration, we have thought it appropriate to illustrate the variations in these quantities by means of some curves which show, partly their dependency on the height for fixed values of ω and partly their dependency on ω for fixed values of the height. As basis for these curves we have taken the »atmosphere« denoted by *F*, and the values given for the conductivities and for changes in the dielectric constant corresponding to 1 electron per c c, while in a single case — namely Fig. 7 — there is further included the influence of 1 ion per c c for $\omega = 0$.

The reason we have included the influence of the ions only in this particular case is the following:

For an ion $h_{ion} = 180$ and thus for $\omega \geqq 10^5$ we have with sufficient approximation

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$$\sigma_{\mathrm{I\,ion}} = \sigma_{\mathrm{II\,ion}} = \sigma_{\perp\,\mathrm{ion}} = \sigma_{0\,\mathrm{ion}} = \frac{e^2}{m_{\mathrm{ion}} c^2} \cdot \frac{\nu_{\mathrm{ion}}}{\nu_{\mathrm{ion}}^2 + \omega^2}, \quad \text{(e. m. u.)} \tag{60}$$

and

$$\Delta_{\mathrm{I}} \varepsilon_{\mathrm{ion}} = \Delta_{\mathrm{II}} \varepsilon_{\mathrm{ion}} = \Delta_{\perp} \varepsilon_{\mathrm{ion}} = \Delta_0 \varepsilon_{\mathrm{ion}} = 4\pi \frac{e^2}{m_{\mathrm{ion}}} \cdot \frac{1}{\nu_{\mathrm{ion}}^2 + \omega^2}, \tag{61}$$

where σ_0 is the conductivity and $\Delta_0 \varepsilon$ the change in the dielectric constant in the absence of a magnetic field.

For all radio-frequencies we can thus with sufficient approximation disregard the influence of the terrestrial magnetic field on the part of the conductivity and the change in dielectric constant, which is caused by the ions.

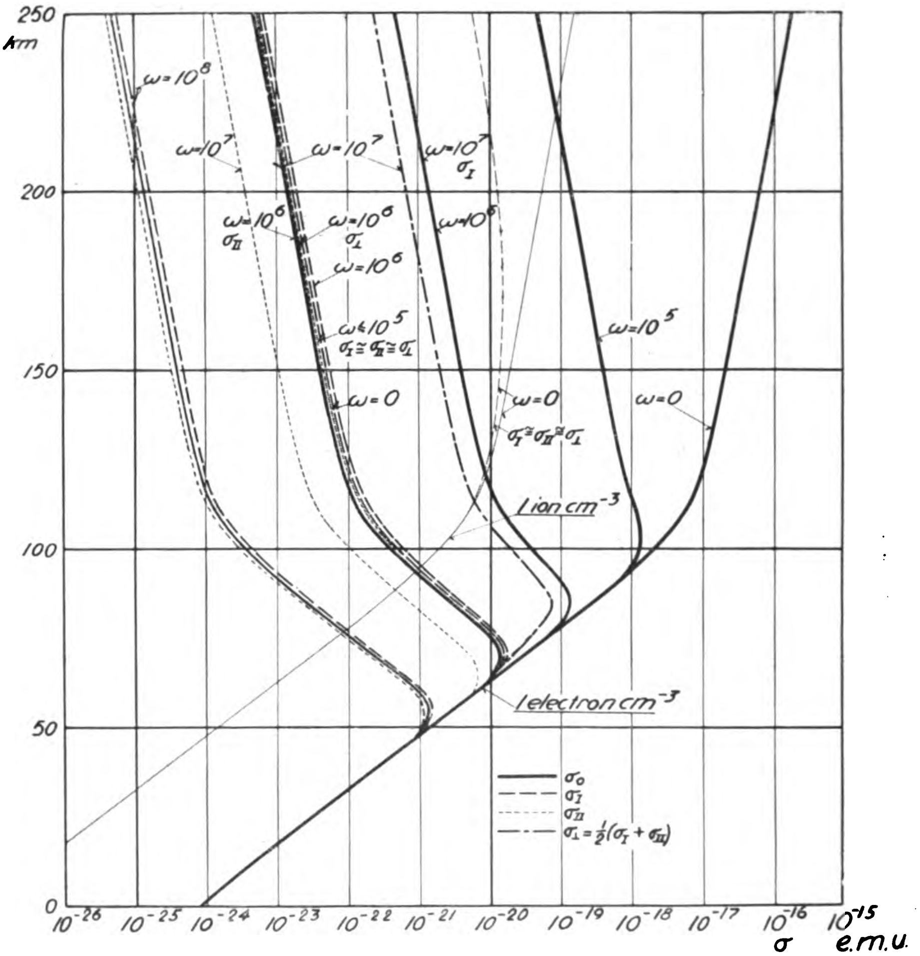

Fig. VII. 7. The Conductivities σ_0, σ_{I}, σ_{II}, and $\sigma_{\perp}$ for 1 Electron per c c (respectively for 1 Ion per c c) as a function of the Altitude h and for $\omega = 0$, 10^5, 10^6, 10^7, and 10^8 (»Atmosphere«: F).

It will be quite different when the frequency approaches zero, *i. e.* if we go over to direct current. In that case we have according to (45 f):

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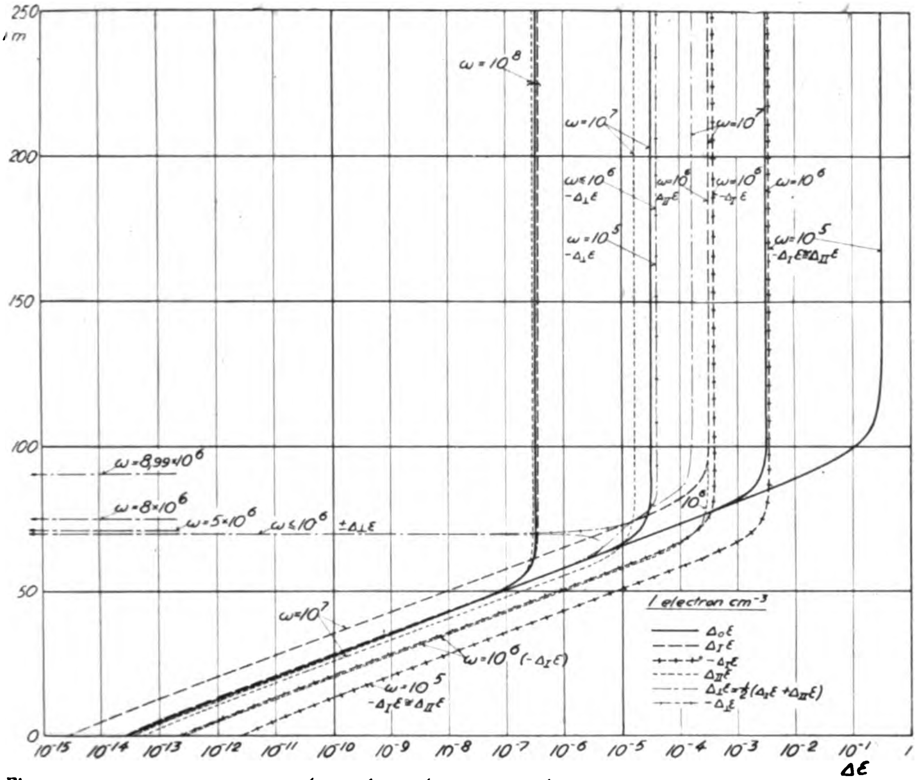

Fig. VII. 8. The Changes $\Delta_0\varepsilon$, $\Delta_{I}\varepsilon$, $\Delta_{II}\varepsilon$, and $\Delta_{\perp}\varepsilon$ in the Dielectric Constant for 1 Electron per c c as a function of the Altitude h and for $\omega = 10^5$, 10^6, 10^7 and 10^8 (»Atmosphere«: F).

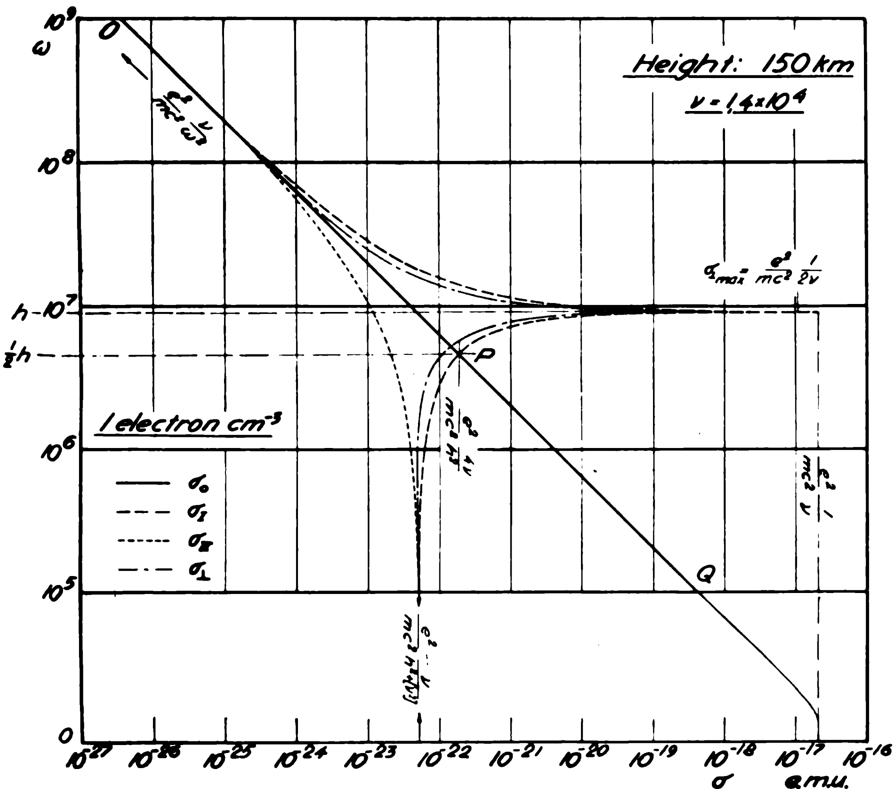

Fig. VII. 9. The Conductivities σ_0, σ_{I}, σ_{II}, and $\sigma_{\perp}$ for 1 Electron per c c as a function of ω for $h = 150$ km (»Atmosphere«: F; $\nu = 1.4 \cdot 10^4$).

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$$\sigma_{\mathrm{I\,ion}} = \sigma_{\mathrm{II\,ion}} = \sigma_{\perp\mathrm{ion}} = \frac{e^2}{m_{\mathrm{ion}} c^2} \cdot \frac{\nu_{\mathrm{ion}}}{\nu_{\mathrm{ion}}^2 + h^2}, \quad (\omega = 0) \tag{62}$$

while

$$\sigma_{0\,\mathrm{ion}\,(\omega=0)} = \frac{e^2}{m_{\mathrm{ion}} c^2} \cdot \frac{1}{\nu_{\mathrm{ion}}}. \tag{63}$$

These values are plotted in Fig. 7 both for 1 electron per c c and for 1 ion per c c. As far as the ion is concerned the conductivity in a magnetic field

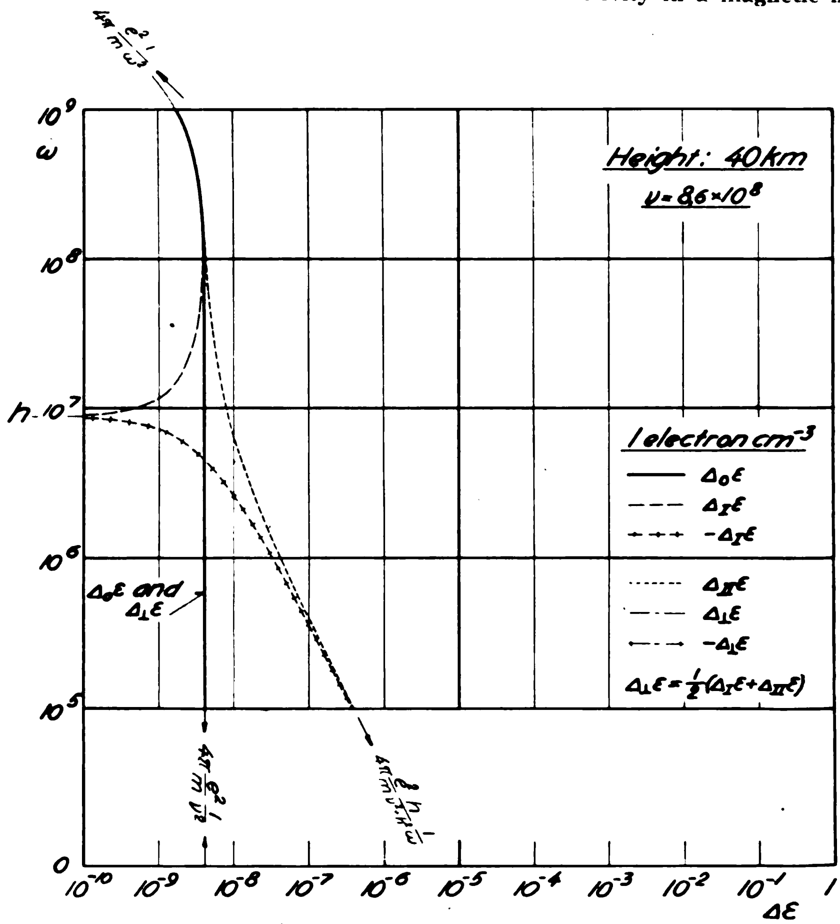

Fig. VII. 10. The Changes $\Delta_0\varepsilon$, $\Delta_{\mathrm{I}}\varepsilon$, $\Delta_{\mathrm{II}}\varepsilon$, and $\Delta_{\perp}\varepsilon$ in the Dielectric Constant for 1 Electron per c c as a function of ω for $h = 40$ km (»Atmosphere«: F; $\nu = 8.60 \cdot 10^8$).

differs appreciably from the conductivity without this magnetic field, only at great heights, while in case of the electron the conductivity in a magnetic field already at a height of 100 km is more than 1000 times smaller than the conductivity without the magnetic field and at a height of 150 km about 10^6 times smaller. The terrestrial magnetic field thus has an exceedingly large influence upon the conductivity of the upper atmosphere for those slowly varying currents which, at least partly, cause the variations in the terrestrial magnetic field.

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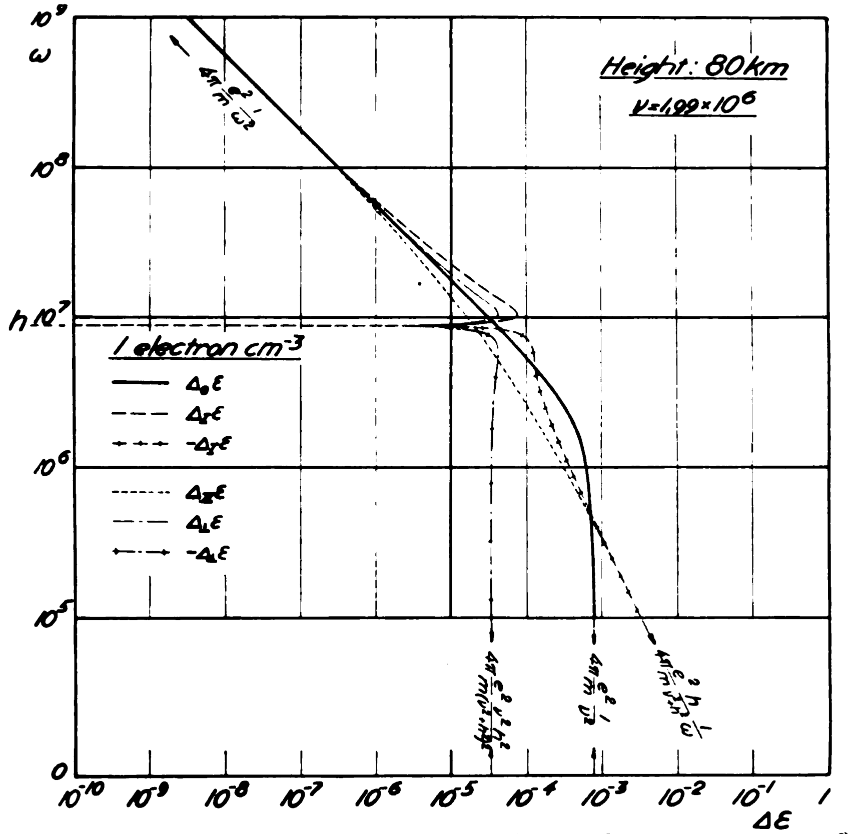

Fig. VII. 11. Same as Fig. 10 except for $h = 80$ km. (›Atmosphere‹: F; $\nu = 1.99 \cdot 10^6$).

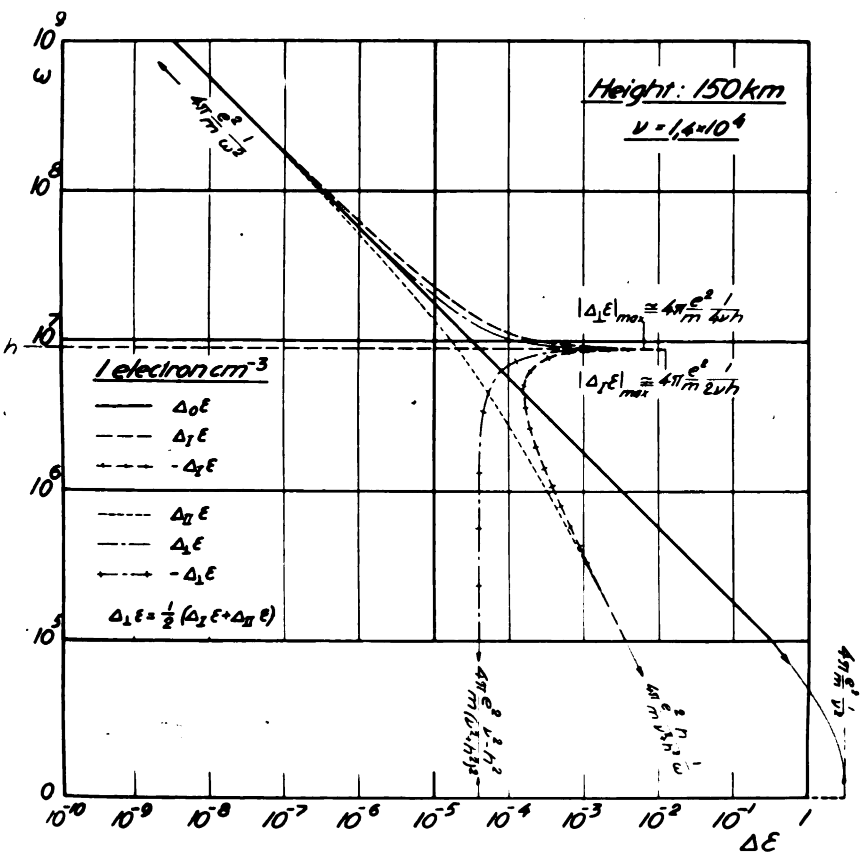

Fig. VII. 12. Same as Fig. 10 except for $h = 150$ km. (›Atmosphere‹: F; $\nu = 1.4 \cdot 10^4$).

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The values of $\Delta_{\mathrm{I}}\varepsilon$, $\Delta_{\mathrm{II}}\varepsilon$ and $\Delta_{\perp}\varepsilon$ for $\omega \to 0$ have evidently no particular interest and we shall therefore give them no further consideration.

Fig. 7 shows the dependency of the conductivities on the height for $\omega = 0$, 10^5, 10^6, 10^7, and 10^8 while Fig. 8 shows the values of $\Delta_0\varepsilon$, $\Delta_{\mathrm{I}}\varepsilon$, $\Delta_{\mathrm{II}}\varepsilon$ and $\Delta_{\perp}\varepsilon$ for the same frequencies with exception of $\omega = 0$.

Fig. 9 shows the dependency of the conductivities on the frequency for $h = 150$ km. The curve will be almost the same, however, for the interval $100 < h < 250$ km. For $\omega = \frac{1}{2}h$ we get

$$\sigma_{\mathrm{I}} = \sigma_0 = \frac{\mathrm{e}^2}{\mathrm{mc}^2} \cdot \frac{4\nu}{\mathrm{h}^2 + 4\nu^2} \tag{64}$$

corresponding to the point P in Fig. 9. For $\nu \ll h$ this formula may be reduced to

$$\sigma_{\mathrm{I}} = \sigma_0 = \frac{\mathrm{e}^2}{\mathrm{mc}^2} \cdot \frac{4\nu}{\mathrm{h}^2}. \tag{65}$$

For $\omega = h$ we have

$$\sigma_{\mathrm{I}\,(\omega=\mathrm{h})} = \sigma_{0\,(\omega=0)} = \frac{\mathrm{e}^2}{\mathrm{mc}^2} \cdot \frac{1}{\nu}. \tag{66}$$

We always have $\sigma_{\mathrm{II}} < \sigma_0$, and for $\omega \to 0$ the equations (62) and (63) will apply.

The Figs. 10—12 show the change in the dielectric constant as a function of ω for $h = 40$, 80 and 150 km respectively.

We always have $\Delta_{\mathrm{II}}\varepsilon > 0$ and $\Delta_{\mathrm{I}}\varepsilon \to 0$ for $\omega \to h$. Further we have $\Delta_{\mathrm{I}}\varepsilon = -\Delta_{\mathrm{II}}\varepsilon$ for $\nu^2 + \omega^2 = h^2$ to which corresponds $\Delta_{\perp}\varepsilon = 0$. For $\omega \leqq 10^6$ this case occurs at a height of about 70 km, see Fig. 8. In this figure those heights where $\Delta_{\perp}\varepsilon = 0$ for $\omega = 5 \cdot 10^6$, $8 \cdot 10^6$ and $8.99 \cdot 10^6$ are also shown. For $\omega \to 9 \cdot 10^6$ this height approaches infinity.

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CHAPTER VIII.

THE PROPAGATION OF A PLANE WAVE IN A HOMOGENEOUS CONDUCTING MEDIUM, AND REFLECTION FROM A PLANE SURFACE BOUNDING TWO DIFFERENT MEDIA.

1. *The Propagation of a Plane Wave in a Homogeneous Conducting Medium.*

If the wave is moving in the direction of the X-axis we have, using the same symbols as in Chapter III:

$$\left.\begin{aligned} &E_x = 0, \quad E_y = 0, \quad H_x = 0, \quad H_z = 0 \quad \text{and further} \\ &E_z = a e^{-sx + j\omega t} \quad \text{and} \quad H_y = j\,\frac{s}{\omega} \cdot a \cdot e^{-sx + j\omega t}, \end{aligned}\right\} \tag{1}$$

which expressions satisfy the equations (I) and (II) in Chapter III when the value of s is determined in such a manner that:

$$s^2 = j\omega\left(4\pi\sigma + j\omega\,\frac{\varepsilon}{c^2}\right) = -\frac{\omega^2}{c^2}\,\varepsilon + j\,4\pi\omega\sigma, \tag{2}$$

where, as usual, c indicates the velocity of light in free space, σ the conductivity of the medium in e. m. u., and ε its dielectric constant.

Writing

$$s = \gamma_0 + j\,\delta_0 \tag{3}$$

we have

$$\gamma_0^2 = \frac{\omega^2}{c^2}\left\{\sqrt{\frac{\varepsilon^2}{4} + \left(2\pi c^2\,\frac{\sigma}{\omega}\right)^2} - \frac{\varepsilon}{2}\right\}, \tag{4}$$

and

$$\delta_0^2 = \frac{\omega^2}{c^2}\left\{\sqrt{\frac{\varepsilon^2}{4} + \left(2\pi c^2\,\frac{\sigma}{\omega}\right)^2} + \frac{\varepsilon}{2}\right\}. \tag{5}$$

Here γ_0 is the attenuation constant, since the amplitude of the waves during their propagation is attenuated at the rate of $e^{-\gamma_0 x}$, while

$$v = \frac{\omega}{\delta_0} \tag{6}$$

is the phase velocity of the wave in the direction of the X-axis.

The propagation of the wave in the medium is depending on both the attenuation constant γ_0, and the index of refraction n which is determined by

$$n = \frac{c}{v} = \frac{c}{\omega}\,\delta_0 = \sqrt{\frac{\varepsilon}{2} + \sqrt{\frac{\varepsilon^2}{4} + \left(2\pi c^2\,\frac{\sigma}{\omega}\right)^2}}. \tag{7}$$

The equations (4) and (6) result in

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$$\gamma_0 n = 2\pi c \sigma. \tag{6a}$$

Most authors, in treating this problem, substitute for equation (7) the far simpler:

$$n = \sqrt{\varepsilon}. \tag{7a}$$

This equation, however, cannot at all be applied in the frequently occurring and very important case of ε decreasing towards zero or even assuming large negative values. *Baker* and *Rice*[1], however, have deduced in a very interesting manner a formula corresponding exactly to (7), but in their paper which treats only the propagation of short waves ($\lambda < 60$ m) they nevertheless use only the simple formula (7a).

In order to facilitate the synopsis of the variation of γ_0 with σ and ε we have shown, in Fig. 1, the dependency of $\frac{1}{\gamma_0}$ on σ and ω for the case $\varepsilon = 1$. Fig. 2 is perfectly analogous to Fig. 1, but gives the value of $\frac{1}{\gamma_0}$ for a series of values of ε ranging from $\varepsilon = 10$ to $\varepsilon = 0$. Fig. 3 gives the value of $\frac{\omega}{\gamma_0}$ as a function of $\frac{\sigma}{\omega}$ for a series of positive and negative values of ε. Finally, Figs. 4 and 5[2] show nomograms for the determination of $\frac{1}{\gamma_0}$, the first one on the basis of an approximate formula for $\frac{1}{\gamma_0}$ which is useful within quite wide limits. Regarding these limits and the use of the nomograms, further information is given in the text below the figures.

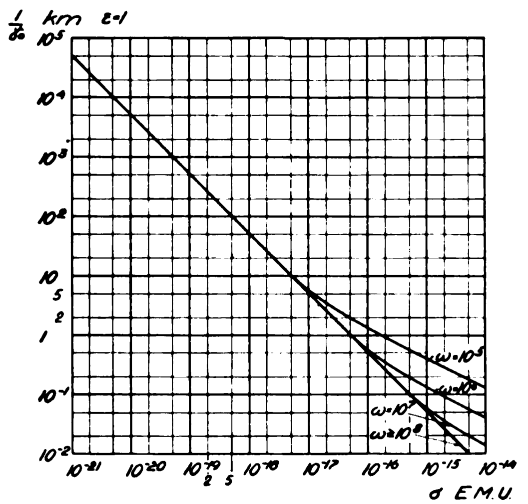

Fig. VIII. 1. The Value of $\frac{1}{\gamma_0}$ as a Function of σ for $\varepsilon = 1$.

In corresponding manner Fig. 6 gives the value of the refractive index as a function of $\Delta\varepsilon$ and $\frac{\sigma}{\omega}$, and computed on the basis of equation (7). It will easily be seen that for

$$\Delta\varepsilon = 1 - \varepsilon \gtreqless \left(2\pi c^2 \frac{\sigma}{\omega}\right)^2 \quad \text{we have} \quad n \lesseqgtr 1. \tag{7b}$$

An examination of equation (7) shows that n is always positive. For $\varepsilon \gg 1$ or $2\pi c^2 \frac{\sigma}{\omega} \gg 1$, and simultaneously $\varepsilon > 0$, we find $n \gg 1$. For very high negative values of ε, n approaches zero.

While Fig. 6 gives a rather good general picture of the value of n as a function of $\Delta\varepsilon$ and $\frac{\sigma}{\omega}$, this diagram is not convenient for a numerical deter-

[1] *W. G. Baker* and *C. W. Rice:* Refraction of Short Radio Waves in the Upper Atmosphere, p. 77 of Complete Paper (A. I. E. E. 1926).

[2] See the Appendix p. 2 and 3.

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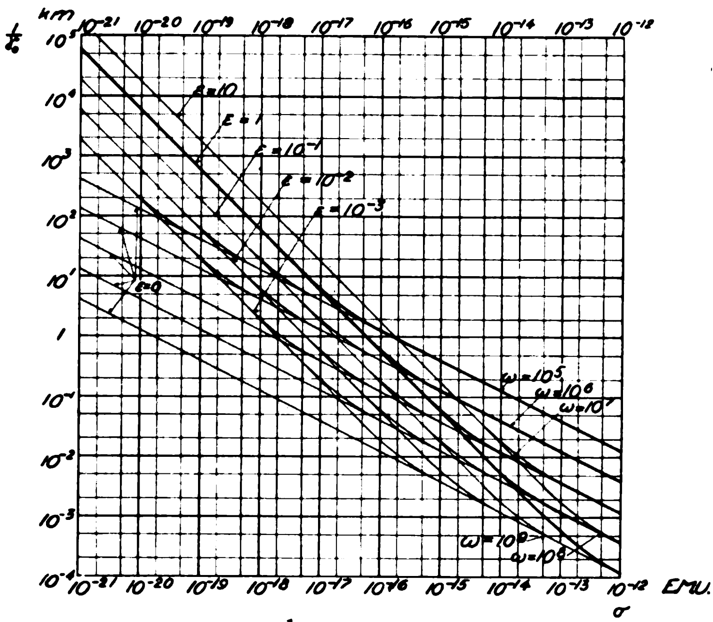

Fig. VIII. 2. The Value of $\frac{1}{\gamma_0}$ as a Function of σ for various positive values of ε.

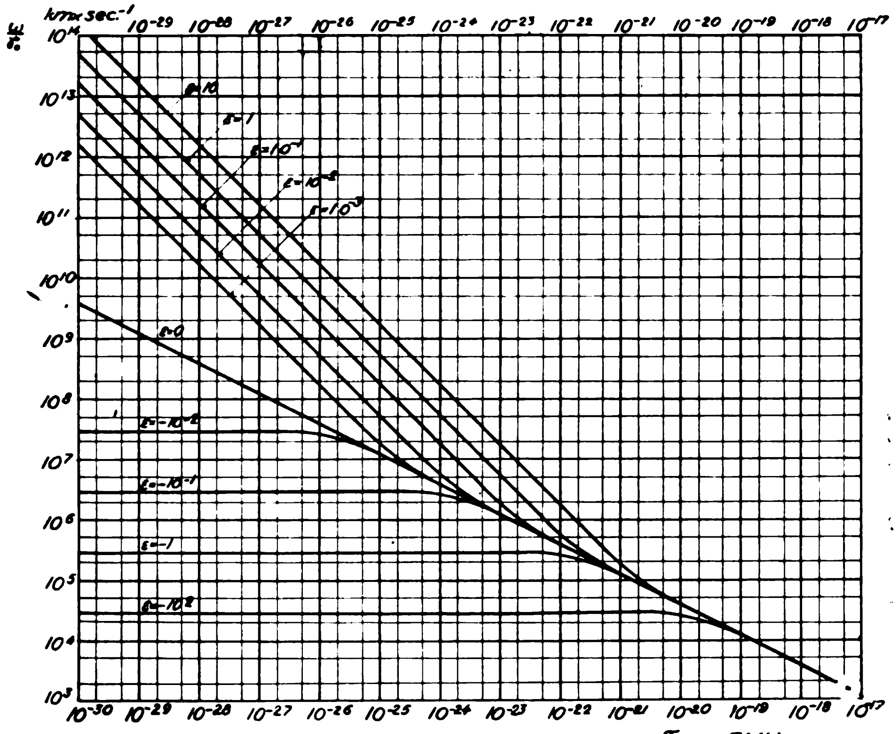

Fig. VIII. 3. The Value of $\frac{\omega}{\gamma_0}$ as a Function of $\frac{\sigma}{\omega}$ for a series of positive and negative values of ε.

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mination of n for which purpose the nomograms shown in Figs. 7 and 8[1] may be used to advantage. These nomograms are constructed on the basis of the equation:

$$n^4 - \varepsilon n^2 - \left(2\pi c^2 \frac{\sigma}{\omega}\right)^2 = 0, \tag{7c}$$

which is simply a transformation of equation (7). Directions for using the nomograms will be found in the text below the figures.

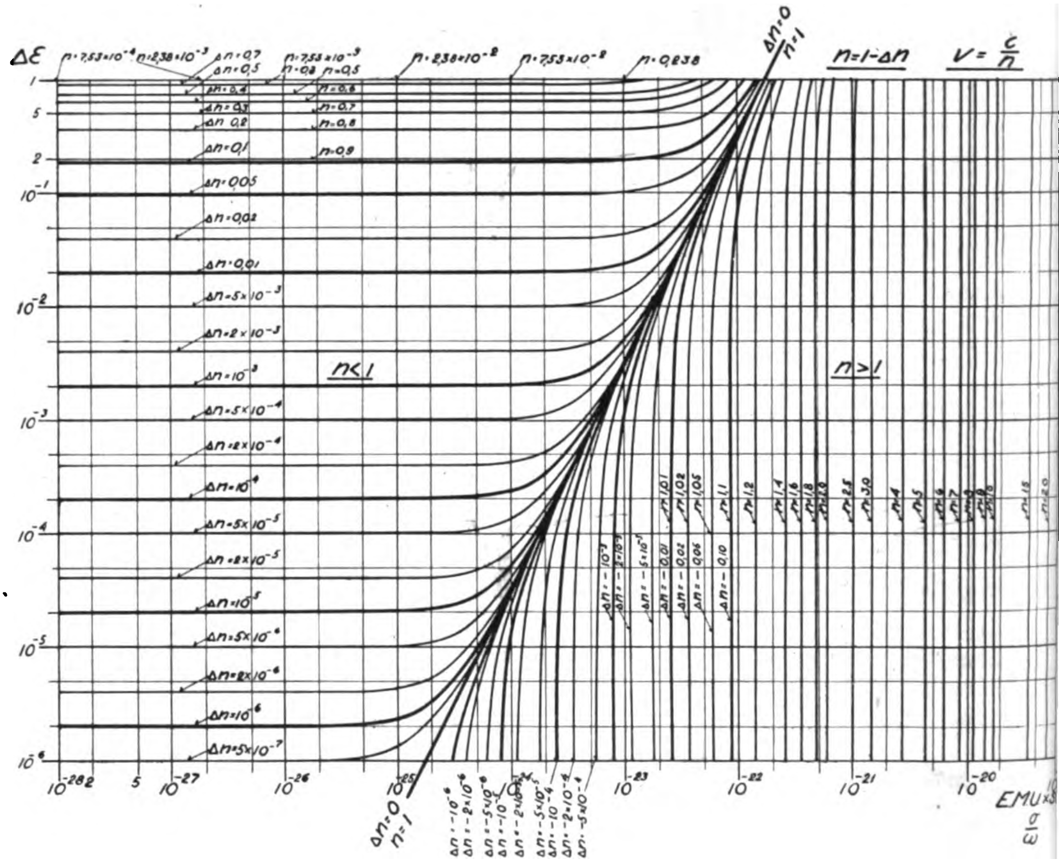

Fig. VIII. 6. Values of $n = 1 - \Delta n$ as a Function of $\Delta\varepsilon = 1 - \varepsilon$ and of $\frac{\sigma}{\omega}$.

The above determinations of γ_0 and n apply in any case whether or not the influence of the magnetic field be taken into consideration. One has merely in each individual case to use the proper values of σ and ε.

Although in general it will be necessary to use, as above, the complete formulas (4) and (7) for a correct determination of the attenuation constant and the index of refraction, a clearer view of the subject may be attained by means of some of the following approximate formulas for γ_0 and n, giving in a number of definite cases a rather good approximation for the determination of these quantities. This summary is divided into two main cases, the conduc-

[1] See the Appendix p. 4 and 5.

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tivity and the dielectric constant being determined (a) without and (b) with regard to the earth's magnetic field, *i. e.* corresponding to Chapter VI and Chapter VII, respectively. In addition a few general approximation formulas will first be given.

General Approximation Formulas for γ_0 and n.

For $\varepsilon > 0$ and $\sigma \ll \frac{\omega\varepsilon}{4\pi c^2}$ we have:

$$\gamma_0 = \frac{2\pi c\sigma}{\sqrt{\varepsilon}}\,[\mathrm{cm}^{-1}] = \frac{6\pi\sigma}{\sqrt{\varepsilon}}\cdot 10^{15}\,[\mathrm{km}^{-1}]; \qquad n = \sqrt{\varepsilon}\left(1 + \frac{2\pi^2 c^4\sigma^2}{\varepsilon^2\omega^2}\right) \cong \sqrt{\varepsilon}. \tag{8}$$ [1]

For $\varepsilon < 0$ and $\sigma \ll \frac{\omega\,|\varepsilon|}{4\pi c^2}$:

$$\gamma_0 = \frac{\omega}{c}\sqrt{|\varepsilon|}\,[\mathrm{cm}^{-1}] = \frac{\omega}{3}\sqrt{|\varepsilon|}\cdot 10^{-5}\,[\mathrm{km}^{-1}]; \qquad n = \frac{2\pi c^2\sigma}{\omega\sqrt{|\varepsilon|}}, \tag{9}$$

and finally for $|\varepsilon| \ll 4\pi c^2 \frac{\sigma}{\omega}$:

$$\gamma_0 = \sqrt{2\pi\omega\sigma}\,[\mathrm{cm}^{-1}] = 10^5\sqrt{2\pi\omega\sigma}\,[\mathrm{km}^{-1}]; \qquad n = c\sqrt{2\pi\frac{\sigma}{\omega}}. \tag{10}$$

(a) Conductivity and Dielectric Constant determined without considering the influence of the earth's magnetic field.

If N is the number of electrons or ions per unit volume, the equations (21) and (36) in Chapter VI result in:

$$\sigma = N\frac{e^2}{mc^2}\cdot\frac{\nu}{\nu^2+\omega^2}\,[\mathrm{e.\,m.\,u.}]; \quad \varepsilon = 1 - N4\pi\frac{e^2}{m}\frac{1}{\nu^2+\omega^2}. \tag{11}$$

Inserting this value of σ into the expression for γ_0 in (8) we have:

$$\gamma_0 = N\frac{2\pi e^2}{mc\sqrt{\varepsilon}}\cdot\frac{\nu}{\nu^2+\omega^2}\,[\mathrm{cm}^{-1}]; \qquad n_0 = \sqrt{\varepsilon} \tag{8a}$$

which formula for γ_0 agrees with the one found by *Baker* and *Rice*[2], while *Nichols* and *Schelleng*[3] have $n = \sqrt{\varepsilon}$ in the numerator in stead of in the denominator, a difference which may be considerable when the value of n differs essentially from unity.

(b) Conductivity and Dielectric Constant determined while taking into consideration the influence of the earth's magnetic field.

If the electric field vector of the wave is parallel to the magnetic, the latter will have no influence on σ and ε, the values of which are determined by the equations (11).

It will thus not be necessary to consider this case further. In accordance with the equations (27), (28), (32), (33), (44) and (45) in Chapter VII we find for the other cases:

---

[1] Sir *Joseph Larmor*: Phil. Mag. (6). Vol. 48, p. 1025—1036. 1924.

[2] l. c., equation (50).

[3] l. c., p. 229.

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$$\sigma_{\mathrm{I}} = \mathrm{N}\,\frac{\mathrm{e}^2}{\mathrm{mc}^2}\cdot\frac{\nu}{\nu^2+(\omega-\mathrm{h})^2}\,[\mathrm{e.m.u.}]; \qquad \varepsilon_{\mathrm{I}} = 1 - \mathrm{N}4\pi\,\frac{\mathrm{e}^2}{\mathrm{m}}\cdot\frac{\omega-\mathrm{h}}{\omega\,[\nu^2+(\omega-\mathrm{h})^2]}; \tag{12}$$

$$\sigma_{\mathrm{II}} = \mathrm{N}\,\frac{\mathrm{e}^2}{\mathrm{mc}^2}\cdot\frac{\nu}{\nu^2+(\omega+\mathrm{h})^2}\,[\mathrm{e.m.u.}]; \qquad \varepsilon_{\mathrm{II}} = 1 - \mathrm{N}4\pi\,\frac{\mathrm{e}^2}{\mathrm{m}}\cdot\frac{\omega+\mathrm{h}}{\omega\,[\nu^2+(\omega+\mathrm{h})^2]}; \tag{13}$$

$$\sigma_{\perp} = \mathrm{N}\,\frac{\mathrm{e}^2}{\mathrm{mc}^2}\cdot\frac{\nu\,(\nu^2+\omega^2+\mathrm{h}^2)}{(\nu^2+\omega^2+\mathrm{h}^2)^2-4\omega^2\mathrm{h}^2}\,[\mathrm{e.m.u.}]; \quad \varepsilon_{\perp} = 1-\mathrm{N}4\pi\,\frac{\mathrm{e}^2}{\mathrm{m}}\cdot\frac{\nu^2+\omega^2-\mathrm{h}^2}{(\nu^2+\omega^2+\mathrm{h}^2)^2-4\omega^2\mathrm{h}^2}. \tag{14}$$

For $\varepsilon > 0$ and $\sigma \ll \frac{\omega\varepsilon}{4\pi c^2}$ we find the corresponding expressions for $\gamma_{\mathrm{I}}, \gamma_{\mathrm{II}}$ and $\gamma_{\perp}$:

$$\gamma_{\mathrm{I}} = \mathrm{N}\cdot\frac{2\pi\,\mathrm{e}^2}{\mathrm{mc}\sqrt{\varepsilon_{\mathrm{I}}}}\cdot\frac{\nu}{\nu^2+(\omega-\mathrm{h})^2}\,[\mathrm{cm}^{-1}]; \qquad \mathrm{n}_{\mathrm{I}} = \sqrt{\varepsilon_{\mathrm{I}}}; \tag{15}$$

$$\gamma_{\mathrm{II}} = \mathrm{N}\cdot\frac{2\pi\,\mathrm{e}^2}{\mathrm{mc}\sqrt{\varepsilon_{\mathrm{II}}}}\cdot\frac{\nu}{\nu^2+(\omega+\mathrm{h})^2}\,[\mathrm{cm}^{-1}]; \qquad \mathrm{n}_{\mathrm{II}} = \sqrt{\varepsilon_{\mathrm{II}}}; \tag{16}$$

$$\gamma_{\perp} = \mathrm{N}\,\frac{2\pi\,\mathrm{e}^2}{\mathrm{mc}\sqrt{\varepsilon_{\perp}}}\cdot\frac{\nu\,(\nu^2+\omega^2+\mathrm{h}^2)}{(\nu^2+\omega^2+\mathrm{h}^2)^2-4\omega^2\mathrm{h}^2}\,\left[\mathrm{cm}^{-1}\right]; \qquad \mathrm{n}_{\perp} = \sqrt{\varepsilon_{\perp}}. \tag{17}$$

Corresponding formulas for γ are presumably given only by *Nichols* and *Schelleng*[1], and the values given by these authors differ from those set forth above partly by the refractive index being moved from the denominator to the numerator, and partly by ν being omitted in the denominator of γ_{II}, whereafter $\gamma_{\perp}$ is assumed to be equal to $\frac{1}{2}(\gamma_{\mathrm{I}}+\gamma_{\mathrm{II}})$. It should be remembered, however, that the formulas (8), (8a) and (15) to (17) are only approximately correct, since they are based on the assumption that $\varepsilon > 0$ and $\sigma \ll \frac{\omega\varepsilon}{4\pi c^2}$, and these formulas may therefore in many cases and at important points, prove entirely inapplicable.

It will still be necessary to make a few remarks concerning the determination of the refractive index. In the calculation of the values of n as used by *Nichols* and *Schelleng*[2], *Taylor* and *Hulburt*[3] and others, two approximations have been used. In the first place the electrons have been assumed to move perfectly freely, corresponding to ν being assumed equal to zero throughout in our formulas for ε, viz. (11), (12), (13) and (14). In the vicinity of the magnetic resonance frequency, $\omega = \mathrm{h}$, this approximation cannot be used, and for heights below 70 or 100 km it is also poor. In the second place the approximation $\mathrm{n} = \sqrt{\varepsilon}$ is used instead of the accurate formula (7). But in numerous cases, and especially when the waves concerned are not very short, n is essentially different from $\sqrt{\varepsilon}$.

2. *Reflection at a Plane Surface Bounding two Homogeneous Media.*

The electric constants for the two media meeting at the boundary surface are conductivity σ_0 and dielectric constant ε_0 for one medium, and σ and ε for the other. The incident and reflected waves are supposed to travel in the medium σ_0, ε_0 (see Fig. 9).

[1] l. c., p. 230 and 231.

[2] l. c., equations (6), (7) and (8).

[3] l. c., equations (2), (3), (5) and (6).

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The direction of motion A_i of the incident plane wave lies in the plane of incidence XZ, while the XY-plane coincides with the boundary surface. A_i forms the angle of incidence φ with the normal Z to the surface. The reflected wave A_r similarly forms the angle φ with the Z-axis, while the refracted wave A_t forms the angle ψ with the Z-axis. For conducting media, however, ψ will generally be a complex quantity, so that the refracted ray does not have any constant direction in the same manner as the incident and reflected rays.

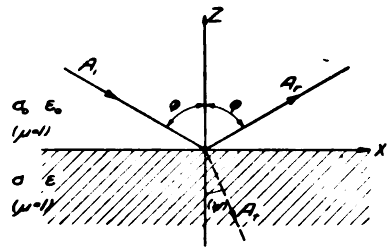

Fig. VIII. 9. Reflection from the Boundary Surface XY. (The Y-axis is perdendicular to the plane of incidence XZ, and its positive direction is towards the plane of the paper).

The problem may be divided into two main cases. In the first case the direction of the electric force lies in the plane of incidence and, consequently, the magnetic force is perpendicular to this plane. On the contrary, in the second case, the magnetic force lies in the plane of incidence, and the electric force is perpendicular to this plane. We indicate the two cases by I and II, respectively.

Case I, the electric force acting in the plane of incidence and the magnetic force at right angle to this plane.

The equations of the incident, reflected and refracted waves may be given the following form:

$$
\left.\begin{array}{ll}
A_i:\left\{\begin{array}{l}
H_{yi}=j\dfrac{s_0}{\omega}\,a_{Ii}\cdot e^{s_0(-x\sin\varphi+z\cos\varphi)+j\omega t}\\
E_{xi}=a_{Ii}\cdot\cos\varphi\cdot e^{s_0(-x\sin\varphi+z\cos\varphi)+j\omega t}\\
E_{zi}=a_{Ii}\cdot\sin\varphi\cdot e^{s_0(-x\sin\varphi+z\cos\varphi)+j\omega t}
\end{array}\right. & \\
A_r:\left\{\begin{array}{l}
H_{yr}=-j\dfrac{s_0}{\omega}\,a_{Ir}\cdot e^{-s_0(x\sin\varphi+z\cos\varphi)+j\omega t}\\
E_{xr}=a_{Ir}\cdot\cos\varphi\cdot e^{-s_0(x\sin\varphi+z\cos\varphi)+j\omega t}\\
E_{zr}=-a_{Ir}\cdot\sin\varphi\cdot e^{-s_0(x\sin\varphi+z\cos\varphi)+j\omega t}
\end{array}\right\}, & \begin{array}{l} s_0^2=j\omega\left(4\pi\sigma_0+j\omega\dfrac{\varepsilon_0}{c^2}\right)\\ \quad=-p_0+jq_0.\end{array}\\
A_t:\left\{\begin{array}{l}
H_{yt}=j\dfrac{s}{\omega}\,a_{It}\cdot e^{s(-x\sin\psi+z\cos\psi)+j\omega t}\\
E_{xt}=a_{It}\cdot\cos\psi\cdot e^{s(-x\sin\psi+z\cos\psi)+j\omega t}\\
E_{zt}=a_{It}\cdot\sin\psi\cdot e^{s(-x\sin\psi+z\cos\psi)+j\omega t}
\end{array}\right\}, & \begin{array}{l} s^2=j\omega\left(4\pi\sigma+j\omega\dfrac{\varepsilon}{c^2}\right)\\ \quad=-p+jq.\end{array}
\end{array}\right\}\quad (I')
$$

These expressions will satisfy the differential equations (I) and (II) in Chapter III, which equations in the present case are reduced to

$$
\left.\begin{array}{l}
-\dfrac{\partial H_y}{\partial t}=\dfrac{\partial E_x}{\partial z}-\dfrac{\partial E_z}{\partial x},\\
4\pi\sigma E_x+\dfrac{\varepsilon}{c^2}\cdot\dfrac{\partial E_x}{\partial t}=-\dfrac{\partial H_y}{\partial z},\\
4\pi\sigma E_z+\dfrac{\varepsilon}{c^2}\cdot\dfrac{\partial E_z}{\partial t}=\dfrac{\partial H_y}{\partial x}.
\end{array}\right\}\quad (I')
$$

The quantities a_{Ir}, a_{It} and ψ, being generally all complex, have to be determined in such a manner that at the boundary surface:

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$$H_{yi} + H_{yr} = H_{yt} \quad \text{and} \quad E_{xi} + E_{xr} = E_{xt}, \tag{I''}$$

and finally the factors of x in the exponents of the expressions for A_i, A_r and A_t must be the same

Consequently we have:

$$s_0 \sin\varphi = s \sin\psi. \tag{18}$$

The equations (I'') result further in:

$$s_0(a_{Ii} - a_{Ir}) = s a_{It}; \quad (a_{Ii} + a_{Ir}) \cos\varphi = a_{It} \cos\psi, \tag{19}$$

and equations (18) and (19) result in:

$$\frac{a_{Ir}}{a_{Ii}} = \frac{\sin 2\psi - \sin 2\varphi}{\sin 2\psi + \sin 2\varphi} = \frac{\operatorname{tg}(\psi - \varphi)}{\operatorname{tg}(\psi + \varphi)} = \frac{s_0\sqrt{s^2 - s_0^2 \sin^2\varphi} - s^2 \cos\varphi}{s_0\sqrt{s^2 - s_0^2 \sin^2\varphi} + s^2 \cos\varphi}, \tag{20}$$

and

$$\frac{a_{It}}{a_{Ii}} = \frac{2 s_0 s \cos\varphi}{s_0\sqrt{s^2 - s_0^2 \sin^2\varphi} + s^2 \cos\varphi}. \tag{21}$$

Case II, the electric force at right angle to the plane of incidence, the magnetic force parallel to this plane. If we put:

$$\left.\begin{array}{l}
\left.\begin{array}{l}
A_i: \begin{cases} E_{yi} = a_{IIi} \cdot e^{s_0(-x\sin\varphi + z\cos\varphi) + j\omega t} \\ H_{xi} = \dfrac{s_0}{j\omega} a_{IIi} \cdot \cos\varphi \cdot e^{s_0(-x\sin\varphi + z\cos\varphi) + j\omega t} \\ H_{zi} = \dfrac{s_0}{j\omega} a_{IIi} \cdot \sin\varphi \cdot e^{s_0(-x\sin\varphi + z\cos\varphi) + j\omega t} \end{cases} \\
A_r: \begin{cases} E_{yr} = a_{IIr} \cdot e^{-s_0(x\sin\varphi + z\cos\varphi) + j\omega t} \\ H_{xr} = -\dfrac{s_0}{j\omega} a_{IIr} \cdot \cos\varphi \cdot e^{-s_0(x\sin\varphi + z\cos\varphi) + j\omega t} \\ H_{zr} = \dfrac{s_0}{j\omega} a_{IIr} \cdot \sin\varphi \cdot e^{-s_0(x\sin\varphi + z\cos\varphi) + j\omega t} \end{cases}
\end{array}\right\}, \quad s_0^2 = -p_0 + jq_0. \\
\left.A_t: \begin{cases} E_{yt} = a_{IIt} \cdot e^{s(-x\sin\psi + z\cos\psi) + j\omega t} \\ H_{xt} = \dfrac{s}{j\omega} a_{IIt} \cdot \cos\psi \cdot e^{s(-x\sin\psi + z\cos\psi) + j\omega t} \\ H_{zt} = \dfrac{s}{j\omega} a_{IIt} \cdot \sin\psi \cdot e^{s(-x\sin\psi + z\cos\psi) + j\omega t} \end{cases}\right\}, \quad s^2 = -p + jq,
\end{array}\right\} \tag{II}$$

then these expressions will satisfy the Maxwell equations which in the present case are reduced to:

$$\frac{\partial H_x}{\partial t} = \frac{\partial E_y}{\partial z}; \quad -\frac{\partial H_z}{dt} = \frac{\partial E_y}{\partial x}; \quad 4\pi\sigma E_y + \frac{\varepsilon}{c^2}\frac{\partial E_y}{\partial t} = \frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x}. \tag{II'}$$

The quantities a_{IIr}, a_{IIt} and ψ, which in general all are complex, have to be determined in such a manner that at the boundary surface we have:

$$E_{yi} + E_{yr} = E_{yt}; \quad H_{xi} + H_{xr} = H_{xt}; \quad H_{zi} + H_{zr} = H_{zt}, \quad (\mu = \mu_0 = 1), \tag{II''}$$

and finally the factors of x in the exponents in the expressions for A_i, A_r and A_t must be alike. This last condition, as in case I, requires:

$$s_0 \sin\varphi = s \sin\psi. \tag{22}$$

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The equations (II″) lead to:

$$\left.\begin{aligned} a_{IIi} + a_{IIr} = a_{IIt};\quad (a_{IIi} - a_{IIr})\, s_0 \cos\varphi = a_{IIt}\cdot s\cdot\cos\psi,\\ (a_{IIi} + a_{IIr})\, s_0 \sin\varphi = a_{IIt} s\cdot\sin\psi.\end{aligned}\right\} \tag{23}$$

The last of the equations (23) may be deduced from the first by means of equation (22). The equations (22) and (23) will further be satisfied by:

$$\frac{a_{IIr}}{a_{IIi}} = \frac{s_0\cos\varphi - s\cos\psi}{s_0\cos\varphi + s\cos\psi} = \frac{\operatorname{tg}\psi - \operatorname{tg}\varphi}{\operatorname{tg}\psi + \operatorname{tg}\varphi} = \frac{\sin(\psi-\varphi)}{\sin(\psi+\varphi)} = \frac{s_0\cos\varphi - \sqrt{s^2 - s_0^2\sin^2\varphi}}{s_0\cos\varphi + \sqrt{s^2 - s_0^2\sin^2\varphi}} \tag{24}$$

and

$$\frac{a_{IIt}}{a_{IIi}} = \frac{2 s_0\cos\varphi}{s_0\cos\varphi + \sqrt{s^2 - s_0^2\sin^2\varphi}}. \tag{25}$$

For $\varphi = 0$ we have:

$$\frac{a_{Ir}}{a_{Ii}} = \left(\frac{a_r}{a_i}\right)_I = \left(\frac{a_r}{a_i}\right)_{II} = \frac{s_0 - s}{s_0 + s};\quad \left(\frac{a_t}{a_i}\right)_I = \left(\frac{a_t}{a_i}\right)_{II} = \frac{2 s_0}{s_0 + s}; \tag{26}$$

and for $\varphi = \frac{\pi}{2}$:

$$\left(\frac{a_r}{a_i}\right)_I = 1;\quad \left(\frac{a_r}{a_i}\right)_{II} = -1;\quad \left(\frac{a_t}{a_i}\right)_I = \left(\frac{a_t}{a_i}\right)_{II} = 0. \tag{27}$$

We shall first consider the case of both media being insulating, *i. e.* $\sigma_0 = \sigma = 0$. In that case $s_0 = j\frac{\omega}{c} n_0$ and $s = j\frac{\omega}{c} n$, and the formulas (20), (21), (24) and (25) are reduced to:

$$\left(\frac{a_r}{a_i}\right)_I = \frac{n_0\sqrt{n^2 - n_0^2\sin^2\varphi} - n^2\cos\varphi}{n_0\sqrt{n^2 - n_0^2\sin^2\varphi} + n^2\cos\varphi};\quad \left(\frac{a_t}{a_i}\right)_I = \frac{2 n_0 n\cos\varphi}{n_0\sqrt{n^2 - n_0^2\sin^2\varphi} + n^2\cos\varphi}; \tag{28}$$

$$\left(\frac{a_r}{a_i}\right)_{II} = \frac{n_0\cos\varphi - \sqrt{n^2 - n_0^2\sin^2\varphi}}{n_0\cos\varphi + \sqrt{n^2 - n_0^2\sin^2\varphi}};\quad \left(\frac{a_t}{a_i}\right)_{II} = \frac{2 n_0\cos\varphi}{n_0\cos\varphi + \sqrt{n^2 - n_0^2\sin^2\varphi}}. \tag{29}$$

In order to illustrate the first two formulas in (28) and (29) we have shown, in Fig. 10, the variation of $\left(\frac{a_r}{a_i}\right)_I$ and $\left(\frac{a_r}{a_i}\right)_{II}$ as functions of φ for the two cases:

(1) $n_0 = 1.0,\quad n = 1,5,$
(2) $n_0 = 1.5,\quad n = 1.0.$

For the angle φ_B determined by

$$\operatorname{tg}\varphi_B = \frac{n}{n_0}, \tag{30}$$

we have $a_{Ir} = 0$ both for $n_0 > n$ and for $n_0 < n$.

In the case $n < n_0$ we have $\left|\frac{a_r}{a_i}\right|_I = \left|\frac{a_r}{a_i}\right|_{II} = 1$ for $\varphi > \varphi_L$, where

$$\sin\varphi_L = \frac{n}{n_0}. \tag{31}$$

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φ_B is *Brewster*'s angle for which the reflection equals zero in case the electric force lies in the plane of incidence. φ_L is the critical angle for which total reflection takes place in case $n_0 > n$.

Fig. 10 shows for $a_{Ii} = a_{IIi} = 1$ the values of $|a_{Ir}|$ and $|a_{IIr}|$ and of θ, where

$$a_{Ir} = |a_{Ir}| \cdot e^{j\theta_I} \quad \text{and} \quad a_{IIr} = |a_{IIr}| \cdot e^{j\theta_{II}}.$$

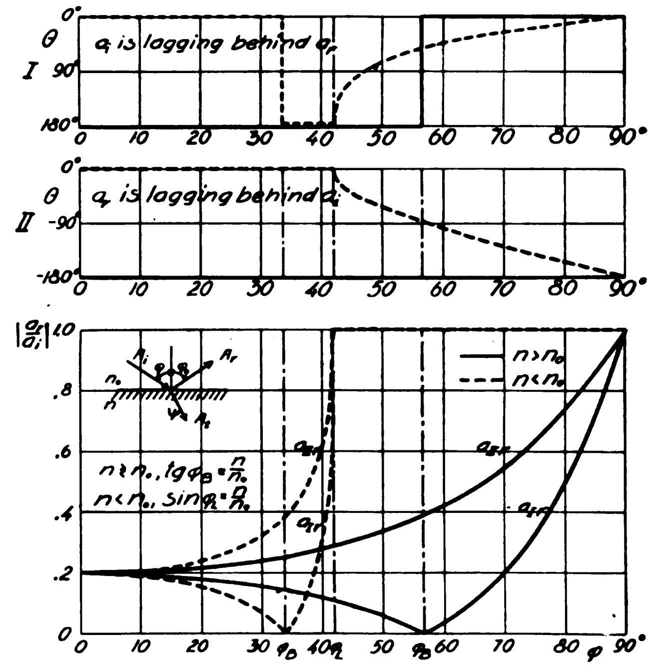

Fig. VIII. 10. For $a_{Ii} = a_{IIi} = 1$, the lower portion of the diagram shows the values of $|a_{Ir}|$ and $|a_{IIr}|$, and the upper portion the values of θ_I and θ_{II}, where $a_{Ir} = |a_{Ir}| e^{j\theta_I}$ and $a_{IIr} = |a_{IIr}| e^{j\theta_{II}}$.

In Case I the Electric Force is Parallel to the Plane of Incidence, and $E_{Irx} = a_{Ir} \cos\varphi$; $E_{Irz} = -a_{Ir} \sin\varphi$, $H_{Irz} = 0$, $H_{Iry} = \frac{n_0}{c} a_{Ir}$.

In Case II the Electric Force is Perpendicular to the Plane of Incidence, and $E_{IIry} = a_{IIr}$, $E_{IIrz} = 0$, $H_{IIrx} = \frac{n_0}{c} a_{IIr} \cos\varphi$, $H_{IIrz} = \frac{n_0}{c} a_{IIr} \cdot \sin\varphi$.

According to (I) and (II) we have

$$p_0 = \frac{\omega^2}{c^2} \varepsilon_0; \quad q_0 = 4\pi\omega\sigma_0, \quad p = \frac{\omega^2}{c^2} \varepsilon; \quad q = 4\pi\omega\sigma. \tag{32}$$

In the following we assume $\sigma_0 = 0$ and $\varepsilon_0 = 1$, so that we get $p_0 = \frac{\omega^2}{c^2}$ and $q_0 = 0$. The equations (20) and (24) may then be written thus:

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$$\frac{a_r}{a_i} = \frac{A + jB}{C + jD} = \frac{AC + BD + j(BC - AD)}{C^2 + D^2} = (r,\ \theta), \tag{33}$$

where, provided $\varepsilon \geqq 0$,

in Case I:

$$\left.\begin{aligned}
A &= -p\cos\varphi + \sqrt{\frac{p}{2\varepsilon}}\sqrt{\sqrt{p^2\left(1-\frac{1}{\varepsilon}\sin^2\varphi\right)^2 + q^2} + p\left(1-\frac{1}{\varepsilon}\sin^2\varphi\right)},\\
B &= q\cos\varphi - \sqrt{\frac{p}{2\varepsilon}}\sqrt{\sqrt{p^2\left(1-\frac{1}{\varepsilon}\sin^2\varphi\right)^2 + q^2} - p\left(1-\frac{1}{\varepsilon}\sin^2\varphi\right)},\\
C &= p\cos\varphi + \sqrt{\frac{p}{2\varepsilon}}\sqrt{\sqrt{p^2\left(1-\frac{1}{\varepsilon}\sin^2\varphi\right)^2 + q^2} + p\left(1-\frac{1}{\varepsilon}\sin^2\varphi\right)},\\
D &= -q\cos\varphi - \sqrt{\frac{p}{2\varepsilon}}\sqrt{\sqrt{p^2\left(1-\frac{1}{\varepsilon}\sin^2\varphi\right)^2 + q^2} - p\left(1-\frac{1}{\varepsilon}\sin^2\varphi\right)},\\
&\text{in Case II:}\\
-A = C &= \sqrt{\frac{1}{2}}\sqrt{\sqrt{p^2\left(1-\frac{1}{\varepsilon}\sin^2\varphi\right)^2 + q^2} - p\left(1-\frac{1}{\varepsilon}\sin^2\varphi\right)},\\
B &= \sqrt{\frac{p}{\varepsilon}}\cos\varphi - \sqrt{\frac{1}{2}}\sqrt{\sqrt{p^2\left(1-\frac{1}{\varepsilon}\sin^2\varphi\right)^2 + q^2} + p\left(1-\frac{1}{\varepsilon}\sin^2\varphi\right)},\\
D &= \sqrt{\frac{p}{\varepsilon}}\cos\varphi + \sqrt{\frac{1}{2}}\sqrt{\sqrt{p^2\left(1-\frac{1}{\varepsilon}\sin^2\varphi\right)^2 + q^2} + p\left(1-\frac{1}{\varepsilon}\sin^2\varphi\right)}.
\end{aligned}\right\} \tag{34}$$

The modulus r of the complex quantity $\frac{a_r}{a_i}$ is determined by:

$$r = \left|\frac{a_r}{a_i}\right| = \sqrt{\frac{A^2 + B^2}{C^2 + D^2}} = \sqrt{\frac{x - y}{x + y}}, \tag{35}$$

where
in Case I:

$$\left.\begin{aligned}
x &= (p^2 + q^2)\cos^2\varphi + \frac{p}{\varepsilon}\sqrt{p^2\left(1-\frac{1}{\varepsilon}\sin^2\varphi\right)^2 + q^2},\\
y &= \sqrt{\frac{2p}{\varepsilon}}\cos\varphi\left\{p\sqrt{\sqrt{p^2\left(1-\frac{1}{\varepsilon}\sin^2\varphi\right)^2 + q^2} + p\left(1-\frac{1}{\varepsilon}\sin^2\varphi\right)}\right.\\
&\quad\left. + q\sqrt{\sqrt{p^2\left(1-\frac{1}{\varepsilon}\sin^2\varphi\right)^2 + q^2} - p\left(1-\frac{1}{\varepsilon}\sin^2\varphi\right)}\right\},
\end{aligned}\right\} \tag{36}$$

in Case II:

$$x = \frac{p}{\varepsilon}\cos^2\varphi + \sqrt{p^2\left(1-\frac{1}{\varepsilon}\sin^2\varphi\right)^2 + q^2},$$

$$y = \sqrt{\frac{2p}{\varepsilon}}\cos\varphi\sqrt{\sqrt{p^2\left(1-\frac{1}{\varepsilon}\sin^2\varphi\right)^2 + q^2} + p\left(1-\frac{1}{\varepsilon}\sin^2\varphi\right)}.$$

The argument θ of $\frac{a_r}{a_i}$ is determined by

$$\theta = \operatorname{arc\,tg}\frac{BC - AD}{AC + BD}, \tag{37}$$

where A, B, C and D have the values determined by (34).

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If $\varphi = 0$ we have

$$\left|\frac{a_r}{a_i}\right|_I^2 = \left|\frac{a_r}{a_i}\right|_{II}^2 = \frac{\sqrt{p^2+q^2} + \frac{1}{\varepsilon}p - \sqrt{\frac{2p}{\varepsilon}}\sqrt{\sqrt{p^2+q^2}+p}}{\sqrt{p^2+q^2} + \frac{1}{\varepsilon}p + \sqrt{\frac{2p}{\varepsilon}}\sqrt{\sqrt{p^2+q^2}+p}}, \tag{38}$$

and

$$\operatorname{tg}\theta_I = \operatorname{tg}\theta_{II} = -\sqrt{\frac{2p}{\varepsilon}}\frac{\sqrt{\sqrt{p^2+q^2}-p}}{\sqrt{p^2+q^2}-\frac{1}{\varepsilon}p} = -\sqrt{\frac{2}{\varepsilon}}\frac{\sqrt{\sqrt{1+\left(\frac{q}{p}\right)^2}-1}}{\sqrt{1+\left(\frac{q}{p}\right)^2}-\frac{1}{\varepsilon}}. \tag{39}$$

For $\varepsilon \geqq \dfrac{1}{\sqrt{1+\frac{q^2}{p^2}}}$ we have that $\operatorname{tg}\theta_{I0} = \operatorname{tg}\theta_{II0} = \operatorname{tg}\theta_0$ are always negative.

For $\frac{q}{p} \to 0$ and for $\frac{q}{p} \to \infty$ we have $\operatorname{tg}\theta_0 \to 0$.

For $\varepsilon \geqq 1$, $\operatorname{tg}\theta_0$ will be minimum for $\frac{q}{p} = \sqrt{\frac{1}{\varepsilon^2} - \frac{4}{\varepsilon} + 3}$, *i.e.* $\sqrt{1+\frac{q^2}{p^2}} = 2 - \frac{1}{\varepsilon}$ and

$$(\operatorname{tg}\theta_0)_{\min} = -\frac{1}{\sqrt{2(\varepsilon-1)}}. \tag{40}$$

From (33) we find easily that for $\varepsilon \geqq 1$ we must have:

$$\left(|a_r|\cdot e^{j\theta}\right)_{\varphi=0} = -a + jb$$

where both a and b are positive constants. Consequently, in these cases, $90^0 < \theta_0 \leqq 180^0$.

For $\varphi = 90^0$ we have:

$$\left|\frac{a_r}{a_i}\right|_I = \left|\frac{a_r}{a_i}\right|_{II} = 1 \quad \text{and} \quad \operatorname{tg}\theta_I = \operatorname{tg}\theta_{II} = 0; \quad \theta_I = 0 \text{ and } \theta_{II} = 180^0.$$

Approximate formulas.

We add below some approximate formulas which may be useful.

For $\frac{q}{p} \gg 1 - \frac{1}{\varepsilon}\sin^2\varphi$ we have approximately,

$$\sqrt{1+\left(\frac{p}{q}\right)^2\left(1-\frac{1}{\varepsilon}\sin^2\varphi\right)^2} = 1 + \frac{1}{2}\left(\frac{p}{q}\right)^2\left(1-\frac{1}{\varepsilon}\sin^2\varphi\right)^2,$$

while for $\frac{q}{p} \ll 1 - \frac{1}{\varepsilon}\sin^2\varphi$ we have

$$\sqrt{1+\left(\frac{p}{q}\right)^2\left(1-\frac{1}{\varepsilon}\sin^2\varphi\right)^2} = \frac{p}{q}\left(1-\frac{1}{\varepsilon}\sin^2\varphi\right) + \frac{\frac{1}{2}\frac{q}{p}}{1-\frac{1}{\varepsilon}\sin^2\varphi}.$$

In the following we shall give only the values of x and y, since by means of these and by using the formula (35) the value of $\left|\frac{a_r}{a_i}\right|$ may be determined without difficulty.

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Case I. The electrical force parallel to the plane of incidence.

A. $\frac{q}{p} \gg 1 - \frac{1}{\varepsilon}\sin^2\varphi$:

$$\left.\begin{aligned} x &= \left(1 + \left(\frac{p}{q}\right)^2\right)\cos^2\varphi + \frac{1}{\varepsilon}\frac{p}{q}, \\ y &= \cos\varphi\sqrt{\frac{2}{\varepsilon}\frac{p}{q}}\left\{1 + \frac{1}{2}\frac{p}{q}\left(1 + \frac{1}{\varepsilon}\sin^2\varphi\right)\right\}. \end{aligned}\right. \tag{41}$$

If further $\frac{p}{q} \ll \varepsilon\cos^2\varphi$, we have

$$\left|\frac{a_r}{a_i}\right| \simeq 1 - \sqrt{\frac{2}{\varepsilon}\frac{p}{q}}\,\frac{\left(1 + \frac{1}{2}\frac{p}{q}\right)\cos\varphi}{\cos^2\varphi + \frac{1}{\varepsilon}\frac{p}{q}} \simeq 1 - \sqrt{\frac{2}{\varepsilon}\frac{p}{q}}\,\frac{1 + \frac{1}{2}\frac{p}{q}}{\cos\varphi}. \tag{42}$$

If $\varepsilon = \varepsilon_0 = 1$, we have

$$\left|\frac{a_r}{a_i}\right| \simeq \left|\frac{q}{p}\,\frac{\cos 2\varphi}{4\cos^2\varphi}\right|. \tag{43}$$

$$\operatorname{tg}\theta = -\sqrt{\frac{2}{\varepsilon}\frac{p}{q}}\cos\varphi\,\frac{1 - \frac{1}{2}\frac{p}{q}\left(1 + \frac{1}{\varepsilon}\right) + \frac{1}{2\varepsilon}\frac{p}{q}\cos^2\varphi}{\left(1 + \left(\frac{p}{q}\right)^2\right)\cos^2\varphi - \frac{1}{\varepsilon}\frac{p}{q}}. \tag{44}$$

B. $\frac{q}{p} \ll 1 - \frac{1}{\varepsilon}\sin^2\varphi$:

$$\left.\begin{aligned} x &= \cos^2\varphi\left(1 + \frac{1}{\varepsilon^2} + \left(\frac{q}{p}\right)^2\right) + \frac{1}{\varepsilon} - \frac{1}{\varepsilon^2}, \\ y &= 2\sqrt{\frac{1}{\varepsilon}}\cos\varphi\sqrt{1 - \frac{1}{\varepsilon}\sin^2\varphi}\cdot\left(1 + \frac{5}{8}\left(\frac{q}{p}\right)^2\right) \simeq 2\sqrt{\frac{1}{\varepsilon}}\cos\varphi\sqrt{1 - \frac{1}{\varepsilon}\sin^2\varphi}, \end{aligned}\right\} \tag{45}$$

$\left[\left|\frac{a_r}{a_i}\right|\right.$ is independent of $\frac{q}{p}$, if $\left.\frac{q}{p} < \text{say } \frac{1}{10}\right]$.

$$\operatorname{tg}\theta = -\cos\varphi\,\frac{q}{p}\sqrt{\frac{1}{\varepsilon}}\,\frac{1 - \frac{2}{\varepsilon}\sin^2\varphi}{\sqrt{1 - \frac{1}{\varepsilon}\sin^2\varphi}}\cdot\frac{1}{\left(1 - \frac{1}{\varepsilon^2} + \left(\frac{q}{p}\right)^2\right)\cos^2\varphi - \left(\frac{1}{\varepsilon} + \frac{1}{\varepsilon^2} + \frac{1}{2}\left(\frac{q}{p}\right)^2\right)}. \tag{46}$$

If further $\varepsilon \gg 1$, we have

$$\operatorname{tg}\theta = -\cos\varphi\,\frac{q}{p}\sqrt{\frac{1}{\varepsilon}}\,\frac{1 - \frac{3}{2}\frac{1}{\varepsilon}\sin^2\varphi}{\left(1 - \frac{1}{\varepsilon^2} + \left(\frac{q}{p}\right)^2\right)\cos^2\varphi - \left(\frac{1}{\varepsilon} + \frac{1}{\varepsilon^2} + \frac{1}{2}\left(\frac{q}{p}\right)^2\right)}. \tag{46a}$$

C. $p = q$, $\sin\varphi \ll \sqrt{\varepsilon}$:

$$\left.\begin{aligned} x &= \cos^2\varphi\left(2 + \sqrt{\frac{1}{2}\frac{1}{\varepsilon^2}}\right) + \frac{\sqrt{2}}{\varepsilon}\left(1 - \frac{1}{2\varepsilon}\right), \\ y &= 3{,}108\sqrt{\frac{1}{\varepsilon}}\cos\varphi\left(1 - \frac{1}{2\sqrt{2}}\frac{1}{\varepsilon}\sin^2\varphi\right), \end{aligned}\right\} \tag{47}$$

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$$\operatorname{tg}\theta = -\frac{2}{\sqrt{\varepsilon}}\sqrt{\sqrt{2}-1}\;\frac{1-\frac{1}{2\sqrt{2}}\frac{1}{\varepsilon}\sin^2\varphi}{\cos^2\varphi\left(2-\sqrt{\frac{1}{2}\frac{1}{\varepsilon^2}}\right)-\frac{\sqrt{2}}{\varepsilon}\left(1-\frac{1}{2\varepsilon}\right)}\cos\varphi. \tag{48}$$

Case II. The electric force perpendicular to the plane of incidence.

A. $\frac{q}{p} \gg 1 - \frac{1}{\varepsilon}\sin^2\varphi$:

$$\left.\begin{aligned} x &= 1 + \frac{1}{\varepsilon}\frac{p}{q}\cos^2\varphi\left[+\frac{1}{2}\left(\frac{p}{q}\right)^2\left(1-\frac{1}{\varepsilon}\sin^2\varphi\right)^2\right], \\ y &= \sqrt{\frac{2}{\varepsilon}\frac{p}{q}}\cos\varphi\left\{1+\frac{1}{2}\frac{p}{q}\left(1-\frac{1}{\varepsilon}\right)+\frac{1}{2\varepsilon}\frac{p}{q}\cos^2\varphi\left[+\frac{1}{8}\left(\frac{p}{q}\right)^2\left(1-\frac{1}{\varepsilon}\sin^2\varphi\right)^2\right]\right\}, \end{aligned}\right\} \tag{49}$$

$$\operatorname{tg}\theta = -\sqrt{\frac{2}{\varepsilon}\frac{p}{q}}\cos\varphi\,\frac{1-\frac{1}{2}\frac{p}{q}\left(1-\frac{1}{\varepsilon}\right)-\frac{1}{2\varepsilon}\frac{p}{q}\cos^2\varphi\left[+\frac{1}{8}\left(\frac{p}{q}\right)^2\left(1-\frac{1}{\varepsilon}\sin^2\varphi\right)^2\right]}{1-\frac{1}{\varepsilon}\frac{p}{q}\cos^2\varphi\left[+\frac{1}{2}\left(\frac{p}{q}\right)^2\left(1-\frac{1}{\varepsilon}\sin^2\varphi\right)^2\right]}. \tag{50}$$

B. $\frac{q}{p} \ll 1 - \frac{1}{\varepsilon}\sin^2\varphi$:

$$\left.\begin{aligned} x &= 1 - \frac{1}{\varepsilon} + \frac{2}{\varepsilon}\cos^2\varphi\left[+\frac{\frac{1}{2}\left(\frac{q}{p}\right)^2}{1-\frac{1}{\varepsilon}\sin^2\varphi}\right], \\ y &= 2\sqrt{\frac{1}{\varepsilon}}\sqrt{1-\frac{1}{\varepsilon}\sin^2\varphi}\,\cos\varphi\left(1+\left[\frac{\frac{1}{8}\left(\frac{q}{p}\right)^2}{\left(1-\frac{1}{\varepsilon}\sin^2\varphi\right)^2}\right]\right), \end{aligned}\right\} \tag{51}$$

$\left[\frac{a_r}{a_i}\right.$ is independent of $\frac{q}{p}$ if $\left.\frac{q}{p} < \text{say } \frac{1}{10}\right]$,

$$\left.\begin{aligned} \operatorname{tg}\theta &= -\sqrt{\frac{1}{\varepsilon}}\frac{q}{p}\cos\varphi\,\frac{1}{\left(1-\frac{1}{\varepsilon}\right)\sqrt{1-\frac{1}{\varepsilon}\sin^2\varphi+\frac{1}{2}\left(\frac{q}{p}\right)^2}} \\ &\cong -\frac{\sqrt{\varepsilon}}{\varepsilon-1}\cdot\frac{q}{p}\,\frac{\cos\varphi}{\sqrt{1-\frac{1}{\varepsilon}\sin^2\varphi}} \end{aligned}\right\} \tag{52}$$

C. $p = q$, $\sin\varphi \ll \sqrt{\varepsilon}$:

$$\left.\begin{aligned} x &= \sqrt{2}\left(1-\frac{1}{2\varepsilon}\right)+\cos^2\varphi\cdot\frac{1}{\varepsilon}\left(1+\sqrt{\frac{1}{2}}\right), \\ y &= \cos\varphi\sqrt{\frac{2\sqrt{2}+2}{\varepsilon}}\left(1-\frac{1}{2\sqrt{2}}\frac{1}{\varepsilon}\sin^2\varphi\right), \end{aligned}\right\} \tag{53}$$

$$\operatorname{tg}\theta = -\sqrt{\frac{2\sqrt{2}-2}{\varepsilon}}\cos\varphi\,\frac{1-\frac{1}{2\sqrt{2}}\frac{1}{\varepsilon}\sin^2\varphi}{\sqrt{2}-\frac{1}{\varepsilon}+\frac{1}{\varepsilon}\left(1-\sqrt{\frac{1}{2}}\right)\sin^2\varphi}. \tag{54}$$

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As examples of the use of these formulas, which correspond to those presented recently by *L. Bouthillon*[1], we have in Figs. 11 to 18 shown the variation of $\frac{a_r}{a_i}$ and θ with φ for a number of different frequencies and for four different kinds of »ground«, the constants of which are given in the table below. This table also gives the values used by *Bouthillon* as well as the limits of the conductivity of English soil as determined by *R. L. Smith-Rose* and *R. H. Barfield*[2].

	Figs. 11 to 18		Figures	*Bouthillon*		Conductivity of English Soil according to *Smith-Rose* and *Barfield*
	σ [e. m. u.]	ε		σ [e. m. u.]	ε	
Sea Water	$1 \cdot 10^{-11}$	80	11 and 12	$1 \cdot 10^{-11}$	80	
Fresh Water	$1 \cdot 10^{-14}$	80	13 and 14	$1 \cdot 10^{-14}$	80	
Wet Soil	$5 \cdot 10^{-14}$	8	15 and 16	$1 \cdot 10^{-13}$	5	$1.5 \cdot 10^{-14} < \sigma < 5.3 \cdot 10^{-13}$
Dry Soil	$1 \cdot 10^{-15}$	4	17 and 18	$1 \cdot 10^{-16}$	2	

If we denote $\left|\frac{a_r}{a_i}\right|$ by r then we always have $r_{II} \geqq r_I$, and r_{II} has its minimum value for $\varphi = 0^0$, and then increases uniformly towards a value equal to unity for $\varphi = 90^0$. For $\lambda > 100$ m r_{II} will only differ essentially from unity in the case of very dry ground. For ocean water and fresh water and for all wave lengths used in practice $r_{II} \cong 1$. r_{II} will only assume values as low as about 0.5 for very short waves and for poorly conducting soil.

The value of r_I, on the other hand, depends far more on φ and for each individual wave length and for certain values of this angle r_I will drop below the value 0.5, or even below 0.025 for certain wave lengths and certain ground conditions. The higher the conductivity of the soil the narrower is the angular field within which r_I assumes these low values. For more detailed information concerning these points, however, reference is made to the said figures.

A receiver situated at or near the surface of the earth is actuated by the incident as well as by the reflected wave and quite complicated conditions may hereby be created, but space does not allow us to enter further into these details. We shall merely call attention to the paper by *Bouthillon*. It may be mentioned, however, that at the earth's surface, the resultant field produced by co-operation between the incident and the reflected wave will be characterized by the magnetic force being horizontal and the electric force being vertical in case of highly conducting ground and $\varphi < 70^0$.

[1] *L. Bouthillon:* Influence du sol et de l'angle d'incidence des ondes électromagnétiques sur le fonctionnement des antennes et des cadres de réception. Journ. de l'École Polytechnique. 2^e s. 25^0 cahier. p. 151—190. 1926.

[2] *R. L. Smith-Rose* and *R. H. Barfield:* Proc. Roy. Soc. (A). Vol. 107. p. 587—601. 1925.

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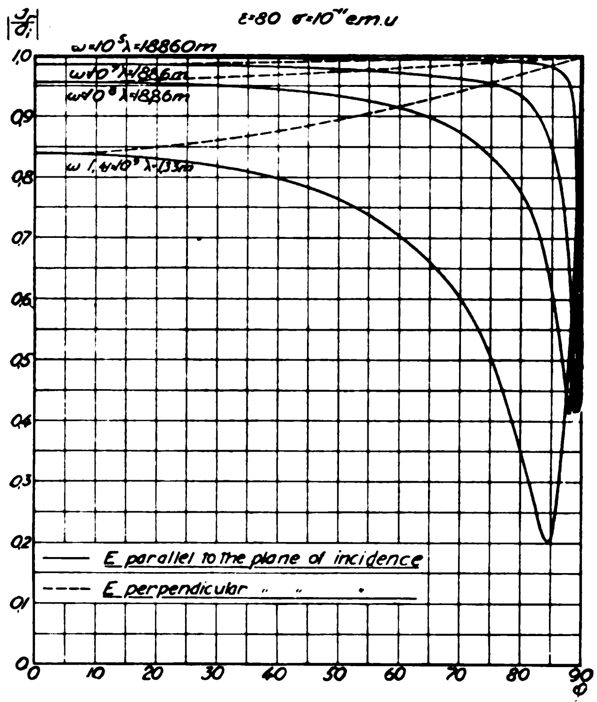

Fig. VIII. 11. Reflection from Ocean Water. Values of $r = \left|\frac{a_r}{a_i}\right|$.

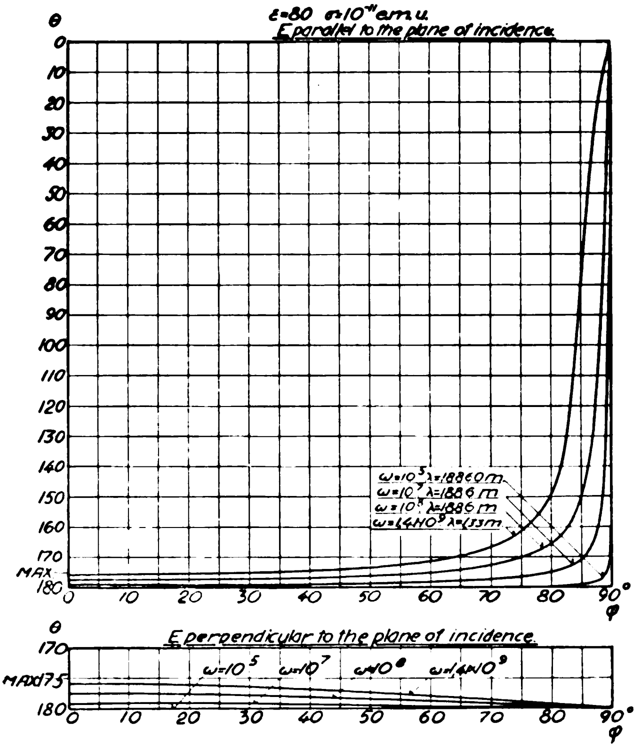

Fig. VIII. 12. Values of θ for Reflection from Ocean Water.

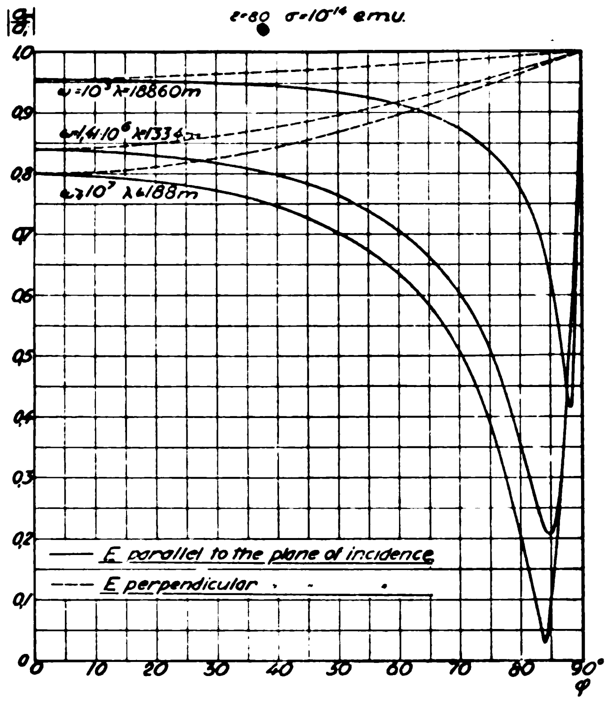

Fig. VIII. 13. Values of r for Reflection at a Fresh Water Surface.

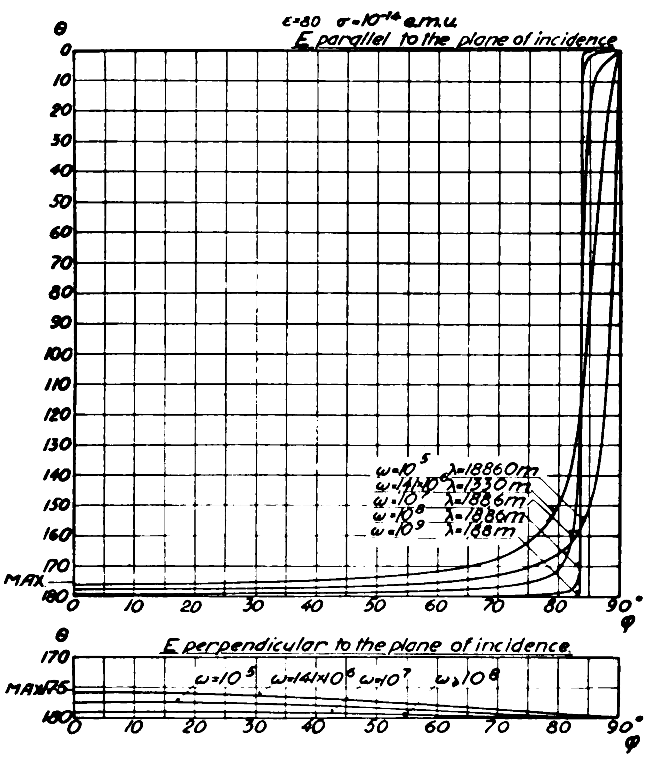

Fig. VIII. 14. Values of θ for Reflection at a Fresh Water Surface.

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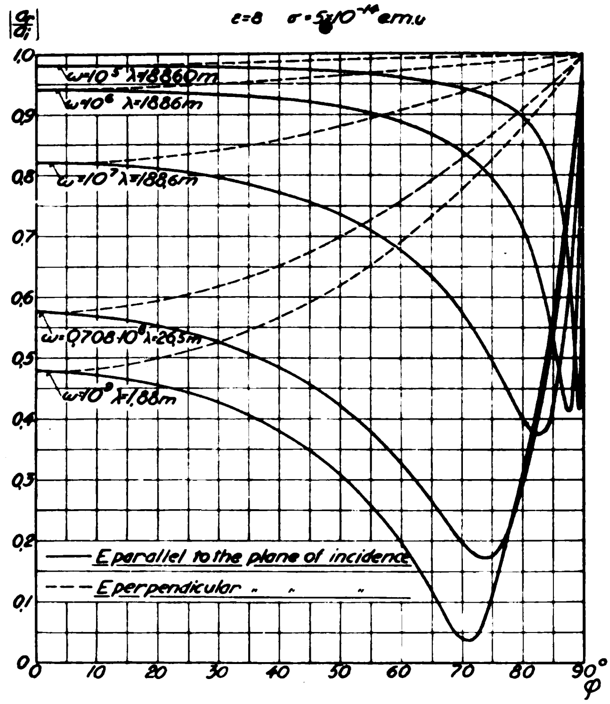

Fig. VIII. 15. Values of r for Reflection at a Surface of Wet Soil.

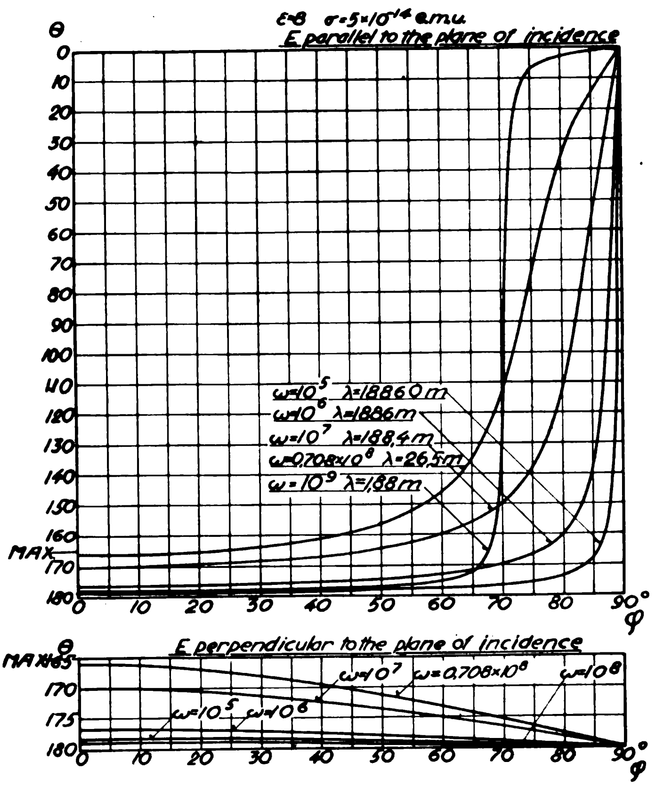

Fig. VIII. 16. Values of θ for Reflection at a Surface of Wet Soil.

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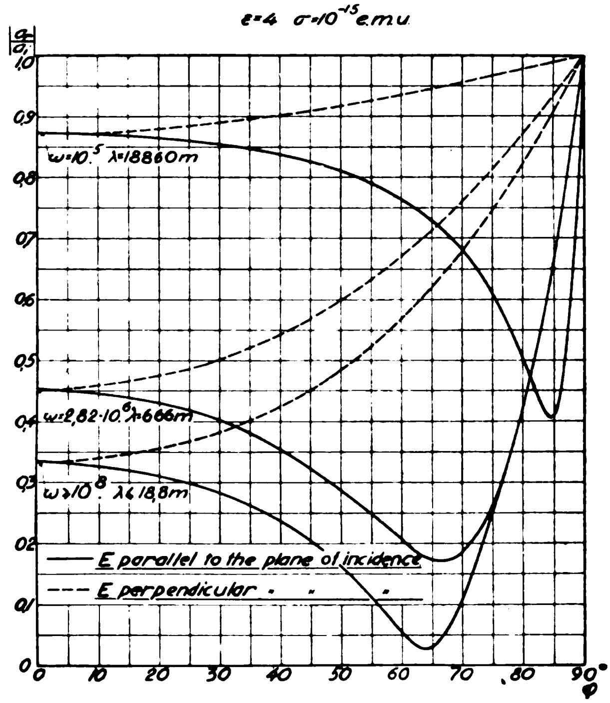

Fig. VIII. 17. Values of r for Reflection at a Surface of Dry Soil.

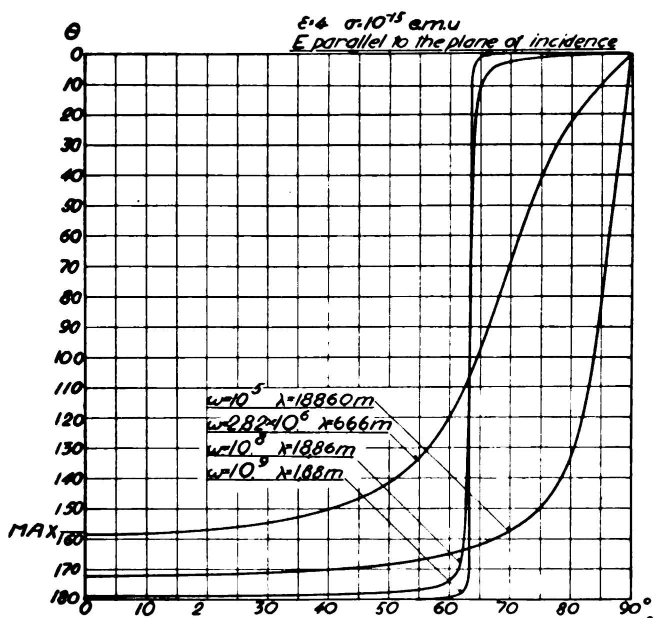

Fig. VIII. 18. Values of θ for Reflection at a Surface of Dry Soil.

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3. *Reflection at a Thin Layer Between two Homogeneous Media.*

The thickness of the separating layer is l cm, its conductivity is σ_0 and the dielectric constant ε_0, while for the medium above and below this layer the

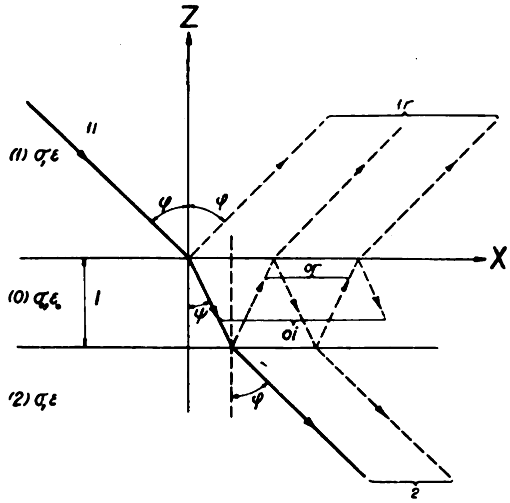

Fig. VIII. 19. The Incident Ray is marked 1_i, the Reflected Ray 1_r, and the Emergent Ray is marked 2.

conductivity is σ and the dielectric constant ε, see Fig. 19. Case I will first be considered.

The incident wave in the upper medium may be written as follows:

$$\left.\begin{aligned} H_{Iyi} &= j\frac{s}{\omega}a_{Ii}\cdot e^{s(-x\sin\varphi+z\cos\varphi)+j\omega t}, \\ E_{Ixi} &= a_{Ii}\cos\varphi\cdot e^{s(-x\sin\varphi+z\cos\varphi)+j\omega t}, \\ E_{Izi} &= a_{Ii}\sin\varphi\cdot e^{s(-x\sin\varphi+z\cos\varphi)+j\omega t}, \end{aligned}\right\} \tag{55}$$

and the reflected wave in the upper medium may[1] be represented by

$$\left.\begin{aligned} H_{Iyr} &= -j\frac{s}{\omega}a_{Ir}\cdot e^{-s(x\sin\varphi+z\cos\varphi)+j\omega t}, \\ E_{Ixr} &= a_{Ir}\cos\varphi\cdot e^{-s(x\sin\varphi+z\cos\varphi)+j\omega t}, \\ E_{Izr} &= -a_{Ir}\sin\varphi\cdot e^{-s(x\sin\varphi+z\cos\varphi)+j\omega t}. \end{aligned}\right\} \tag{56}$$

In the intermediate medium there will exist a downward directed wave:

$$\left.\begin{aligned} H_{0yi} &= j\frac{s_0}{\omega}a_{0i}\cdot e^{s_0(-x\sin\psi+z\cos\psi)+j\omega t}, \\ E_{0xi} &= a_{0i}\cos\psi\cdot e^{s_0(-x\sin\psi+z\cos\psi)+j\omega t}, \\ E_{0zi} &= a_{0i}\sin\psi\cdot e^{s_0(-x\sin\psi+z\cos\psi)+j\omega t}, \end{aligned}\right\} \tag{57}$$

and an upward directed reflected wave:

[1] *Lord Rayleigh:* Theory of Sound. Vol. II. § 271 (Second Ed., 1896) treats the corresponding acoustic problem.

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$$\left.\begin{aligned} H_{0yr} &= -j\frac{s_0}{\omega}\,a_{0r}\cdot e^{-s_0(x\sin\psi+z\cos\psi)+j\omega t}, \\ E_{0xr} &= a_{0r}\cos\psi\cdot e^{-s_0(x\sin\psi+z\cos\psi)+j\omega t}, \\ E_{0zr} &= -\,a_{0r}\sin\psi\cdot e^{-s_0(x\sin\psi+z\cos\psi)+j\omega t}. \end{aligned}\right\} \tag{58}$$

Finally in the lower medium we have a downward directed transmitted wave:

$$\left.\begin{aligned} H_{2y} &= j\frac{s}{\omega}\,a_2\cdot e^{s(-x\sin\varphi+z\cos\varphi)+j\omega t}, \\ E_{2x} &= a_2\cos\varphi\cdot e^{s(-x\sin\varphi+z\cos\varphi)+j\omega t}, \\ E_{2z} &= a_2\sin\varphi\cdot e^{s(-x\sin\varphi+z\cos\varphi)+j\omega t}, \end{aligned}\right\} \tag{59}$$

where

$$s_0^2 = -\frac{\omega^2}{c^2}\varepsilon_0 + j4\pi\omega\sigma_0 = -p_0 + jq_0 \quad\text{and}\quad s^2 = -\frac{\omega^2}{c^2}\varepsilon + j4\pi\omega\sigma = -p + jq. \tag{60}$$

At the upper surface of the separating layer, *i. e.* for $z = 0$, we must have

$$H_{Iyi} + H_{Iyr} = H_{0yi} + H_{0yr};\quad E_{Ixi} + E_{Ixr} = E_{0xi} + E_{0xr}. \tag{61}$$

At the lower boundary of the separating layer, *i. e.* for $z = -l$ we have:

$$H_{0yi} + H_{0yr} = H_{2y};\quad E_{0xi} + E_{0xr} = E_{2x}. \tag{62}$$

Finally, we must necessarily have

$$s\sin\varphi = s_0\sin\psi. \tag{63}$$

The equations (61) and (62) give:

$$\left.\begin{aligned} &s\left(a_{Ii} - a_{Ir}\right) = s_0\left(a_{0i} - a_{0r}\right), && \text{(a)} \\ &\left(a_{Ii} + a_{Ir}\right)\cos\varphi = \left(a_{0i} + a_{0r}\right)\cos\psi, && \text{(b)} \\ &s_0\left(a_{0i}\,e^{-s_0 l\cos\psi} - a_{0r}\,e^{s_0 l\cos\psi}\right) = s\,a_2\,e^{-s l\cos\varphi}, && \text{(c)} \\ &\left(a_{0i}\,e^{-s_0 l\cos\psi} + a_{0r}\,e^{s_0 l\cos\psi}\right)\cos\psi = a_2\cos\varphi\cdot e^{-s l\cos\varphi}. && \text{(d)} \end{aligned}\right\} \tag{64}$$

From the equations (64) the ratio between the amplitudes of the reflected and the incident wave will be found to be determined by:

$$r = \frac{a_{Ir}}{a_{Ii}} = \frac{(\varrho^2-1)(e^{-2\alpha}-1)}{2\varrho(e^{-2\alpha}+1)-(\varrho^2+1)(e^{-2\alpha}-1)} = -\frac{(\varrho^2-1)(e^{2\alpha}-1)}{(\varrho+1)^2 e^{2\alpha}-(\varrho-1)^2}, \tag{65}$$

where $\alpha = s_0 l\cos\psi$ and

$$\varrho_I = \frac{s_0}{s}\cdot\frac{\cos\varphi}{\cos\psi}. \tag{66}$$

For Case II, *i. e.* for the electric force perpendicular to the plane of incidence and for the magnetic force lying in that plane the ratio between the reflected and the incident wave is similarly determined by (65), but the value of the parameter ϱ is in this case:

$$\varrho_{II} = \frac{s_0}{s}\,\frac{\cos\psi}{\cos\varphi}. \tag{67}$$

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For $\varphi = 90^0$ we have in consequence of (66) and (67) $\varrho_1 = 0$ and $\varrho_2 = \infty$ and, correspondingly, according to (65) $r_I = 1$ and $r_{II} = -1$. For $\sigma_0 = \sigma$ and $\varepsilon_0 = \varepsilon$, *i. e.* when the »layer« does not differ from the adjoining media, we have $\varrho = 1$ and $r_I = r_{II} = 0$. The correctness of these results is immediately evident.

For $|\varrho| \gg 1$ and $\alpha \ll 1$ equation (65) is reduced to:

$$r = \frac{a_{Ir}}{a_{Ii}} = -\frac{\frac{\alpha\varrho}{2}}{1 + \frac{\alpha\varrho}{2}}, \tag{68}$$

which for $|\alpha\varrho| \ll 1$ is reduced to

$$r = \frac{a_{Ir}}{a_{Ii}} = -\frac{\alpha\varrho}{2}, \tag{69}$$

and consequently

$$r_I = \left(\frac{a_{Ir}}{a_{Ii}}\right)_I = -\frac{s_0^2}{2s} l \cos\varphi \text{ and } r_{II} = \left(\frac{a_{Ir}}{a_{Ii}}\right)_{II} = -\frac{s_0^2}{2s} \cdot \frac{\cos^2\psi}{\cos\varphi} l. \tag{70}$$

If on the other hand $|\alpha\varrho|$ is not small in comparison with unity, equation (68) has to be used for the determination of r.

According to the approximate formula (70) the value of r_I decreases when φ increases, but, as shown above, it reaches unity for $\varphi = 90^0$. The variation is largely analogous to the variation shown by the curves in Figs. 11, 13, 15 and 17. For $\varphi = 0$ r_{II} is equal to r_I, and the value of $|r_{II}|$ increases uniformly from $|r_{II}|_{\varphi=0}$ to unity corresponding to $\varphi = 90^0$. Therefore, when the question is merely to form an estimate of the conditions, it will be sufficient to determine $(r_I)_{\varphi=0} = (r_{II})_{\varphi=0} = r_0$ and, accordingly, we may write

$$r_0 = -\frac{s_0^2}{2s} l. \tag{71}$$

If in accordance with Chapter V. 7 the charge of the layer is assumed to be 10 e. s. u. cm^{-2} we have correspondingly $N_{12} = \frac{10}{4.77 \cdot 10^{-10}} \cong 2 \cdot 10^{10}$ ions per cm^2. This corresponds to $N = \frac{2 \cdot 10^{10}}{l}$ ions per cm^3. Since each ion per cm^3 increases the conductivity by about 10^{-27}, we find $\sigma_0 = N \cdot 10^{-27} = \frac{2 \cdot 10^{-17}}{l}$, while $\Delta\varepsilon \cong N \cdot 10^{-15} = \frac{2 \cdot 10^{-5}}{l}$, so that we have, with sufficient approximation, $\varepsilon_0 = \varepsilon = 1$.

We make the further assumption $\omega = 3 \cdot 10^5$, and have then:

$$s_0^2 = -10^{-10} + j4\pi \cdot 3 \cdot 10^5 \frac{2 \cdot 10^{-17}}{l} \cong -10^{-10} + j\frac{8 \cdot 10^{-11}}{l},$$

while $s^2 = -10^{-10}$. We assume that l is so small that the last term of the expression for s_0^2 is great in comparison with the first term, in which case the assumption $|\varrho| \gg 1$ is satisfied. We find then

$$|r_0| = \frac{s_0^2}{2s} l = 4 \cdot 10^{-6}.$$

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Apart from the fact that it would be difficult to imagine such a high concentration of the layer as assumed here, it will be seen that the reflection for $\varphi = 0$ is exceedingly slight, even for such long waves as those dealt with here ($\lambda =$ approx. 5600 m). The reflection from such layers will therefore be very slight in any case where φ is not very close to 90^0. Even for $\varphi = 89^0$ the reflection will be very small. Surfaces of discontinuity of the nature here considered can therefore not be of any considerable importance to the propagation of radio waves.

4. *The Reflection of a Ray from a Medium with Continuously Varying Refractive Index.*

When the properties of the medium do not change abruptly, but pass gradually from one value into another one, the above deduced reflection formulas cannot be applied directly, and it will be necessary to consider this point at little further on account of the transition between the ionized and the non-ionized layers of air, and the laws of propagation of electro-magnetic waves passing through these transition layers.

We shall first consider a special case, assuming the medium to be non-conducting and the dielectric constant to depend merely on the x-co-ordinate. The propagation of a plane wave in the direction of the X-axis is then determined by the well known equation

$$\frac{\partial^2 E}{\partial x^2} = \frac{\varepsilon(x)}{c^2} \frac{\partial^2 E}{\partial t^2}. \tag{72}$$

If the units of time and length are chosen k times as great by making

$$X = \frac{1}{k} x, \quad T = \frac{1}{k} t,$$

equation (72) becomes:

$$\frac{\partial^2 E}{\partial X^2} = \frac{\varepsilon(kX)}{c^2} \frac{\partial^2 E}{\partial T^2}. \tag{73}$$

But imagining that simultaneously with the introduction of the new units the ε(x)-curve is stretched k-fold longitudinally, in such a manner that the value of ε corresponding originally to $x = x_0$ will now be found for $x = kx_0$ — in other words substituting $\varepsilon\left(\frac{x}{k}\right)$ for $\varepsilon(x)$ — then $\varepsilon(kX)$ will be changed into $\varepsilon(X)$, the resulting equation being

$$\frac{\partial^2 E}{\partial X^2} = \frac{\varepsilon(X)}{c^2} \frac{\partial^2 E}{\partial T^2}. \tag{74}$$

This shows that the reflection will remain the same in cases where the dielectric constant — or refractive index — of the layers plotted as a function of $\frac{x}{\lambda}$ varies in the same manner.

The reflection will therefore be dependent on the magnitude of the variation of the dielectric constant or the refractive index during the course of one wave-length. If this variation is but slight, the reflection will be quite insigni-

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ficant. That this is really the case has been demonstrated by various authors[1], and may also be illustrated by the following simple reasoning: From a surface at which the refractive index is suddenly altered from n to $n + \Delta n$ and for vertical incidence the ratio of the amplitude of the reflected ray to that of the incident one will be equal to $\frac{\Delta n}{2n}$. The ratio of reflected to incident radiation energy is consequently $\frac{(\Delta n)^2}{4n^2}$. If the variation of the refractive index is infinitely small of the first order the reflected energy thus becomes infinitely small of the second order.

More definite information about this question has been given by *R. Gans*[2] who has shown that in a non-conducting medium in which the refractive index n only depends on x and in which both n and $\frac{dn}{dx}$ vary continuously there will — with the possible exception of total reflection — be no appreciable reflection if

$$\left(\frac{1}{n}\cdot\frac{dn}{dx}\cdot\frac{\lambda}{2\pi}\right)^2 \ll 1 \quad \text{and} \quad \frac{1}{n}\cdot\frac{d^2n}{dx^2}\left(\frac{\lambda}{2\pi}\right)^2 \ll 1. \tag{75}$$

If n varies continuously while the value of $\frac{dn}{dx}$ at $x = x^0$ changes abruptly from $\left(\frac{dn}{dx}\right)_1$ to $\left(\frac{dn}{dx}\right)_2$ the ratio of the energy of the reflected to that of the incident ray will, for normal incidence, be determined by:

$$\frac{S_r}{S_i} = \frac{\lambda^2}{64\pi^2 n^2}\cdot\left[\left(\frac{dn}{dx}\right)_1 - \left(\frac{dn}{dx}\right)_2\right]^2. \tag{76}$$

Formulas (75) and (76) refer to normal incidence. There will probably be considerable difficulties in extending these formulas to an arbitrary angle of incidence φ. We shall therefore only consider the much simpler problem, namely the reflection in a non-conducting medium and at a surface in which the refractive index n abruptly changes from the value $n_0 = 1$ to the value $n = n_0 - \Delta n = 1 - \Delta n$, where $\Delta n \ll 1$. (By abruptly we mean, that this change takes place within a distance which is small in comparison to the wave length, say less than $\frac{1}{4}\lambda$). According to formulas (28) and (29) we have

$$\left.\begin{aligned} \left|\frac{a_r}{a_i}\right|_{\mathrm{I}} &= \left|\frac{\sqrt{\cos^2\varphi - 2\cdot\Delta n} - (1 - 2\cdot\Delta n)\cos\varphi}{\sqrt{\cos^2\varphi - 2\cdot\Delta n} + (1 - 2\cdot\Delta n)\cos\varphi}\right| \\ \text{and} \quad \left|\frac{a_r}{a_i}\right|_{\mathrm{II}} &= \left|\frac{\cos\varphi - \sqrt{\cos^2\varphi - 2\cdot\Delta n}}{\cos\varphi + \sqrt{\cos^2\varphi - 2\cdot\Delta n}}\right|. \end{aligned}\right\} \tag{77}$$

For $\Delta n \ll \cos\varphi$ these formulas reduce to

$$\left|\frac{a_r}{a_i}\right|_{\mathrm{I}} = \left|\frac{2\cos^2\varphi - 1}{2\cos^2\varphi}\right|\cdot\Delta n \quad \text{and} \quad \left|\frac{a_r}{a_i}\right|_{\mathrm{II}} = \frac{1}{2\cos^2\varphi}\cdot\Delta n. \tag{78}$$

[1] See for instance: *E. Gehrcke:* Handbuch der physikalischen Optik. Bd. 1, p. 219. (*R. Straubel:* Dioptrik in Medien mit kontinuierlich variablem Brechungsindex. 1926). *J. Boussinesq:* Compt. rend. 129, p. 794—799, 859—864, 905—911. 1899. *H. Seeliger:* Phys. Zeitschr. 5, p. 237—238. 1904. *R. Gans:* Ann. d. Phys. (IV). Bd. 47, p. 709—736. 1915.

[2] l. c. p. 713.

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For $\varphi \geqq \varphi_L$, where the critical angle φ_L is determined by

$$\sin \varphi_L = 1 - \Delta n, \tag{79}$$

we have

$$\left|\frac{a_r}{a_i}\right|_I = \left|\frac{a_r}{a_i}\right|_{II} = 1. \tag{80}$$

The angle $\beta = \frac{\pi}{2} - \varphi_L$ is determined by

$$\beta = \sqrt{2 \cdot \Delta n} \text{ [radians]} = 81 \sqrt{\Delta n} \text{ [degrees]}. \tag{81}$$

Table 6. **Reflection at a Surface in which the Refractive Index has a very small Discontinuity.**

$\varphi =$	0	30	45	60	75	80	85^0	$\geqq \varphi_L$
$\left\|\frac{a_r}{a_i}\right\|_I =$	$0.5 \cdot \Delta n$	$0.33 \cdot \Delta n$	0	$1 \cdot \Delta n$	$6.5 \cdot \Delta n$	$15.5 \cdot \Delta n$	$65 \cdot \Delta n$	1.0
$\left\|\frac{a_r}{a_i}\right\|_{II} =$	$0.5 \cdot \Delta n$	$0.667 \cdot \Delta n$	$1 \cdot \Delta n$	$2 \cdot \Delta n$	$7.5 \cdot \Delta n$	$16.5 \cdot \Delta n$	$66 \cdot \Delta n$	1.0

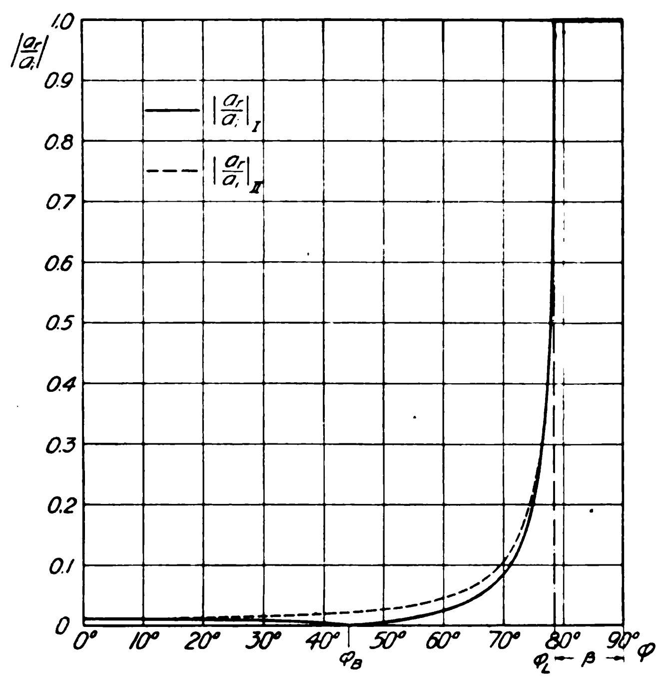

Fig. VIII. 20. The values of $\left|\frac{a_r}{a_i}\right|_I$ and $\left|\frac{a_r}{a_i}\right|_{II}$ as functions of φ for $\Delta n = 0.02$.

We have, in the above table, given the values of $\left|\frac{a_r}{a_i}\right|_I$ and $\left|\frac{a_r}{a_i}\right|_{II}$ for

various values of φ, and in Fig. 20 we have given the values of the same quantities as functions of φ for $\Delta n = 0.02$.

It appears from the table and from Fig. 20 that even very small discontinuities will, especially in case II, cause considerable reflection when the angle of incidence is greater than 60^0. It is, therefore, to be expected that there will be some reflection even from a region in which n varies continuously, and this reflection will be greater in case II than in case I.

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CHAPTER IX.

NUMERICAL VALUES OF THE ELECTRICAL AND OPTICAL PROPERTIES OF THE ATMOSPHERE AS FUNCTIONS OF THE ALTITUDE AND OF THE FREQUENCY OF THE WAVES.

1. *Numerical Values for the Atmosphere F' (and for F).*

In Chapters IV to VIII we have discussed the ionization of the atmosphere and the influence of this ionization on the electrical properties, conductivity and dielectric constant, and on the »optical« properties, refractive index and attenuation constant, of the atmosphere. In this chapter we shall discuss the numerical values of these quantities for the cases which are to be considered in Chapt. XI.

In the said chapter it will be shown that the length of the mean free path of the electrons at high altitudes cannot be smaller than in the »atmosphere« *F* (see Chapt. IV, table 3), but on the contrary must be considerably greater. The air pressure and the mass density at high altitudes cannot reasonably be assumed to be considerably smaller than in the »atmosphere« *F*. We will therefore find it necessary to discuss the question of the mean free path of electrons in an atmosphere consisting mainly of helium. This question has previously been dealt with, but only in a provisional manner, in Chapt. IV, sect. 2.

In Chapt. IV, sect. 2 we assumed the mean free path of the electrons in the atmosphere under consideration to be equal to the free path in a gas having the same pressure and consisting entirely of nitrogen. This is of course only a rough approximation and it will be shown in Chapt. XI that this question is of some importance. Therefore we shall now consider the evidence in this matter a little more closely.

C. Ramsauer[1] and others[2] have shown that the mean free path of slow electrons in the rare gases argon, krypton and xenon attains very high values when the velocity of the electrons is considerably smaller than that corresponding to one volt. The effective cross-sectional area of these molecules at these small velocities of the electrons is many times smaller than their areas as determined by means of the methods of the kinetic theory of gases. With

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[1] *C. Ramsauer:* Ann. d. Phys. Bd. 64, p. 513. 1921; Bd. 66, p. 546. 1921; Bd. 72, p. 345. 1923.

[2] *J. Franck* u. *P. Jordan:* Anregung von Quantensprüngen durch Stösse, Figs. 3 and 4a, p. 20—21 (Berlin 1926), and Handb. d. Physik. Bd. XXIII. Quanten. p. 641—775, especially Figs. 3a, b and 4 and p. 650. (Berlin 1926).

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regard to helium the experimental evidence is not quite so reliable, in so far as it has not been possible to get down to velocities small enough to increase greatly the mean free path of the electron. However, at a velocity corresponding to one volt the curve representing the cross-sectional area of helium shows a marked downward bend[1]. Theoretical considerations point in the same direction[2]. There can hardly be any doubt that the free path of the very slow electrons, with which we are dealing here[3], will be several times as large in helium as in nitrogen. It is, however, impossible on the basis of the available evidence to give a definite value with any degree of certainty; but as we have to decide upon a value in order to carry out the calculations in the following, we assume $l_{el,\,He} = 10\,l_{el,\,N_2}$, which value is probably not very far off the mark.

The following numerical values therefore refer partly to the atmosphere F, assuming the mean free path of the electrons to be the same as if the total air pressure was due to nitrogen and partly to the same atmosphere F under the assumption that the mean free path of the electrons is the same as in a nitrogen atmosphere having a pressure p' determined by

$$p' = 0.1\,p_{\text{rare gases}} + (p - p_{\text{rare gases}}), \tag{1}$$

where $p_{\text{rare gases}}$ is the pressure of the rare gases in the atmosphere F, and accordingly very nearly equal to p_{He}.

In the following we denote the atmosphere F with the free path of the electrons calculated according to formula (1) by F', while the symbol F is retained for the same atmosphere with l_{el} determined according to formula (7) in Chapt. IV. The mean free path of the ions is the same in F and F'.

Fig. IX. 1 (p. 6 in the appendix) shows the air pressure, the mean free paths of electrons and ions, the numbers of collisions for electrons and ions and the recombination constant as functions of the altitude for the »atmospheres« F and F'.

The value of the recombination constant is taken as

$$\alpha = 1.6 \cdot 10^{-6}\,\frac{p}{760} + 4 \cdot 10^{-11}, \tag{1a}$$

a value which (see Chapt. V, sect. 1) is probably a little too great.

Fig. IX. 2 (p. 7 in the appendix) shows for several frequencies the conductivity σ_0 due to one electron or one ion per c c as a function of the altitude while Fig. IX. 3 (p. 8 in the appendix) shows the values of the conductivities σ_I, σ_{II} and $\sigma_\perp$ for one electron per c c. Both figures correspond to the atmosphere F.

Fig. IX. 4 and 5 (p. 7 and 9 in the appendix) show the corresponding values of the dielectric constants $\Delta_0\varepsilon$, $\Delta_I\varepsilon$, $\Delta_{II}\varepsilon$ and $\Delta_\perp\varepsilon$.

In order to determine the ionization of the air we have used the values given in the following table.

Figs. IX. 6 and 7 (p. 10 in the appendix) show the number of electrons and of equivalent ions (see Chapt. V, sect. 12) at altitudes from $h = 0$ to $h = 250$ km for summer and winter, corresponding to the constants given in Table 7.

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[1] *Franck* u. *Jordan:* l. c., Figs. 3b and 4.

[2] *F. Hund:* Z. f. Phys. Bd. 13, p. 241—263. 1923.
M. Born u. *P. Jordan:* Z. f. Phys. Bd. 33, p. 479—50. 1925.

[3] The velocities of the electrons correspond to about 0.04 volts (see page 43).

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Table 7. Ionization Constants. Atmospheres *F* and *F'*.

	Remarks
Water vapour pressure $p_{H_2O} = 0$ at high altitudes	
Recombination constant $\alpha = 1.6\cdot 10^{-6}\cdot\frac{p}{760} + 4\cdot 10^{-11}$	See Chapt. V, sect. 1, Fig. 3 and Fig. IX. 1 (p. 6 in the app.)
Height of »Homogeneous atmosphere«[1] $H = 6.63$ km $\left(\frac{1}{H} = 0.1508\right)$ used in calculating the solar and stellar ionization for altitudes greater than h_m[2]. **	[1] See Chapt. IV formula (2) and table 2, and Chapt. V, sect 3. [2] See Chapt. V formula (58).
Solar Ionization — Summer Midday: Coefficient of absorption $A_{sum} = 5\cdot 10^6$; Constant $\varkappa_{sum} = 1.88\cdot 10^4$; $H' = 1.5H$ for $h < h_m = 114.8$ km	See Chapt. V, sects. 3 and 9. $A_{win} = A_{sum}\cdot\frac{\cos(41^0 - 23\frac{1}{2}^0)}{\cos(41^0 + 23\frac{1}{2}^0)} \simeq 2.2\cdot A_{sum}$.
Solar Ionization — Winter Midday: Coefficient of absorption $A_{win} = 1.1\cdot 10^7$; Constant $\varkappa_{win} = 8.6\cdot 10^3$; $H' = 1.5H$ for $h < h_m = 120.6$ km	$\varkappa_{win} = \frac{\varkappa_{sum}}{2.2}$ *, see formula (60) in Chapt. V.
Stellar Ionization***: Coefficient of absorption $A_{st} = 5\cdot 10^6$; Constant $\varkappa_{st} = 10^{-3}\varkappa_{sum} = 18.8$	See Chapt. V, sect. 10.
In calculating the values of $\Delta_I\varepsilon$, $\Delta_{II}\varepsilon$, $\Delta_\perp\varepsilon$, σ_I, σ_{II}, $\sigma_\perp$, n_I, n_{II}, $n_\perp$, γ_I, γ_{II} and $\gamma_\perp$ the intensity of the magnetic field is assumed to be $H \simeq 0.51$ gauss, corresponding to $h = \frac{e}{mc}H \cong 9\cdot 10^6$	See Chapt. VII formula (11) and p. 103.

* The ratio A_{sum}/A_{win} corresponds to a northern latitude of about 41^0.

** For $h < h_m$ the ionization is taken as $I_h = I_{max}\cdot e^{\left(1-\frac{h_m-h}{H'}-e^{\frac{h_m-h}{H'}}\right)}$; (compare Chapt. V., formula (58) and p. 70—71).

*** In calculating the ionization caused by the very penetrating radiation (see Chapt. V, sect. 5) we have assumed $I = \frac{1}{2} I_{Kolhörster}$, $I_{Kolhörster}$ being determined from equation (62), Chapt. V.

Fig. 8 gives a general view of the density of electrons and of equivalent ions according to Figs. 6 and 7.

Fig. IX. 9 (p. 11 in the appendix) shows $\Delta_0\varepsilon$ as a function of the altitude and the frequency for the atmosphere *F'* and for summer.

Fig. IX. 10 (p. 11 in the appendix) shows, for the interval between $h = 160$ km and $h = 80$ km, the corresponding values of $\Delta_0\varepsilon$, $\Delta_I\varepsilon$, $\Delta_{II}\varepsilon$ and $\Delta_\perp\varepsilon$.

Figs. IX. 11 and 12 (p. 12 and 13 in the appendix) show the corresponding values of σ_0, σ_I, σ_{II} and $\sigma_\perp$.

Fig. IX. 13 (p. 14 in the appendix) shows the values of the refractive index n_0 and of the attenuation constant γ_0 for the atmosphere *F'* and in summer time for the interval from $h = 0$ to $h = 250$ km, while Fig. IX. 14 shows the same

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quantities from $h = 80$ to $h = 160$ km and Fig. IX. 15 for the same interval the corresponding values of n_0, n_{I}, n_{II}, $n_{\perp}$, γ_0, γ_{I}, γ_{II} and $\gamma_{\perp}$.

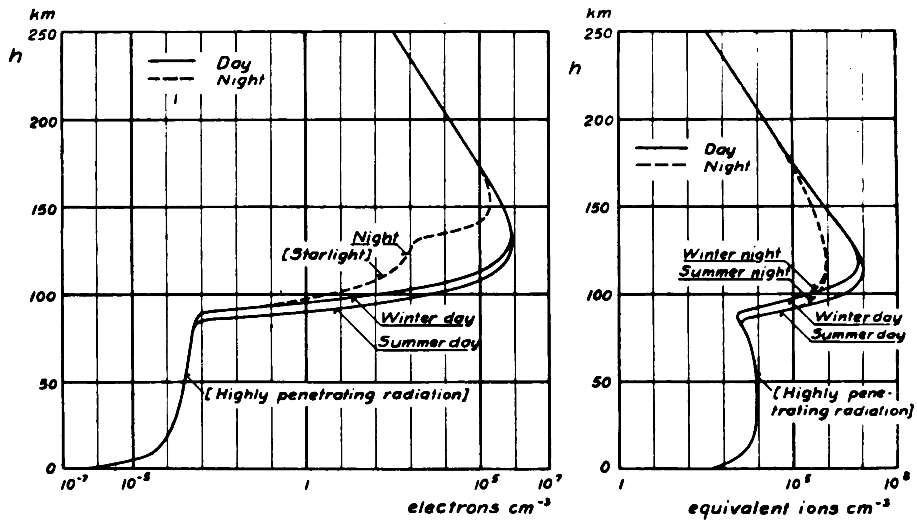

Fig. IX. 8. Densities of electrons and of equivalent ions according to Figs. 6 and 7 by Summer and Winter, Day and Night. (Atmosphere F′).

Figs. IX. 16 and 17 (p. 17 and 18 in the appendix) are exactly similar to Figs. 14 and 15, the only difference being that Figs. 14 and 15 refer to summer and Figs. 16 and 17 to winter conditions.

Fig. IX. 18 (p. 19 in the appendix) shows the values of the refractive index n_0 and the attenuation constant γ_0 for altitudes between $h = 100$ and $h = 150$ km and for the atmosphere F and summer time.

For the atmosphere F' and for summer time Figs. IX. 1—6 and 9—12 give the results of the preliminary calculations necessary to determine the »optical constants» given in Figs. IX. 13—15 for the atmosphere F' and for summer time. For the discussion of the propagation of radio waves we need only the optical constants given in the three last mentioned figures, but for the sake of completeness we have for one single case shown the results of all the intermediate calculations necessary to calculate the final optical constants of the air.

For the atmosphere F' and for winter time we have in Figs. IX. 16 and 17 given the optical constants only between the altitudes 80 and 160 km. At higher and lower altitudes the difference between summer and winter is quite insignificant. For the atmosphere F and for summer time we have, in Fig. IX 18, given the optical constants n_0 and γ_0, only between the altitudes 100 and 150 km. Below 100 km the difference between the atmospheres F and F' is small, and since the atmosphere F, as will be shown later, cannot serve as a basis for a theory of radio wave propagation, there is no necessity for giving the magneto-optical constants for this atmosphere.

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2. *General Remarks. The Conductivity in the Upper Atmosphere and the Solar and Lunar Diurnal Variations of Terrestrial Magnetism.*

Before closing this chapter we shall make just a few remarks regarding the general shape of the curves representing the optical (n_0, γ_0) and magneto-optical constants ($n_{I}, n_{II}, n_{\perp}, \gamma_{I}, \gamma_{II}$ and $\gamma_{\perp}$) and the evidence concerning the conductivity in the upper atmosphere which may be derived from the theory of the solar and lunar diurnal variations in the magnetic field of the earth.

According to Chapt. VIII, formula (6a) we have:

$$\gamma n = 2\pi c\sigma \qquad [\gamma \text{ in } cm^{-1}, \ \sigma \text{ in e. m. u.}] \tag{2}$$

or

$$\gamma n = 6\pi\cdot 10^{15}\cdot\sigma \qquad [\gamma \text{ in } km^{-1}, \ \sigma \text{ in e. m. u.}]. \tag{2a}$$

It follows from these equations that if n decreases rapidly with increasing altitude the same will be the case with $\frac{1}{\gamma}$. An inspection of the (n, h)- and $\left(\frac{1}{\gamma}, h\right)$-curves in Figs. IX. 13 to 18 (p. 14 in the appendix) confirms this result.

For $\Delta\varepsilon = 1$ we have $\varepsilon = 0$ and, therefore, according to equations (4) and (7) in Chapt. VIII:

$$\left.\begin{aligned} n &= \sqrt{2\pi c^2 \frac{\sigma}{\omega}} = c\sqrt{\frac{\sigma}{f}} = 3\cdot 10^{10}\sqrt{\frac{\sigma}{f}} = 10^6\sqrt{3\lambda\sigma} \qquad [\lambda \text{ in m}, \ \sigma \text{ in e. m. u}] \quad (3)\\ \gamma &= \frac{\omega}{c}\, n\,[cm^{-1}] = \frac{\omega}{3}\cdot n\cdot 10^{-5}\,[km^{-1}] = 10^5\sqrt{2\pi\sigma\omega}\,[km^{-1}]\\ &= 2\pi\cdot 10^5\sqrt{f\sigma} = 2\pi\cdot 10^9\sqrt{\frac{3\sigma}{\lambda}}\ [km^{-1};\ \lambda \text{ in m}]. \quad (4) \end{aligned}\right\}\ \begin{aligned}\Delta\varepsilon&=1,\\ \varepsilon&=0.\end{aligned}$$

The values of n determined by the formula (3) are so small for all angular frequencies $\geqq 10^6$ that all incident waves having an angle of incidence greater than a few degrees will be refracted back to the earth, and the corresponding values of $\frac{1}{\gamma}$ are, according to formula (2), relatively small. The waves are therefore strongly attenuated at the altitudes $h_{(\varepsilon=0)}$, for which $\varepsilon = 0$. These altitudes are shown in Fig. IX. 14 (p. 15 in the appendix) and it appears from that figure that the n-curves — for $\omega \geqq 10^6$ — here decrease very rapidly with increasing altitudes and that the corresponding values of n are very small.

A consequence of this is that only very long waves are able to penetrate to such altitudes for which $\Delta\varepsilon > 1$ ($\varepsilon < 0$).

But it also appears from Fig. IX. 14, that $\Delta\varepsilon$ in the daytime can exceed the value $\Delta\varepsilon = 1$, and ε therefore can come down to zero only for $\omega \leqq 5\cdot 10^7$; the corresponding night time frequency limit is $\omega \leqq 2.5\cdot 10^7$ and the altitudes are marked *DL* and *NL* respectively.

For the sake of simplicity we have, in Figs. IX 13 to 18, shown only the lower part of those n curves which bend down almost to zero values, because only this part is of any practical importance for the problem at hand. However, in order to illustrate the characteristic features of the (n, h)-curves we have shown five such curves in Fig. 19.

In Fig. IX. 20 (p. 19 in the appendix) we have for summer time shown the altitude h at which the refractive index has a certain value ($h = 0.9$) as a function of ω. These curves will be discussed later.

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The theoretical work of *A. Schuster*[1] on the diurnal variations of the earth's magnetic field has been continued and extended by others, especially by *S. Chapman*[2]. *Chapman*'s theory of the solar and lunar diurnal variations of

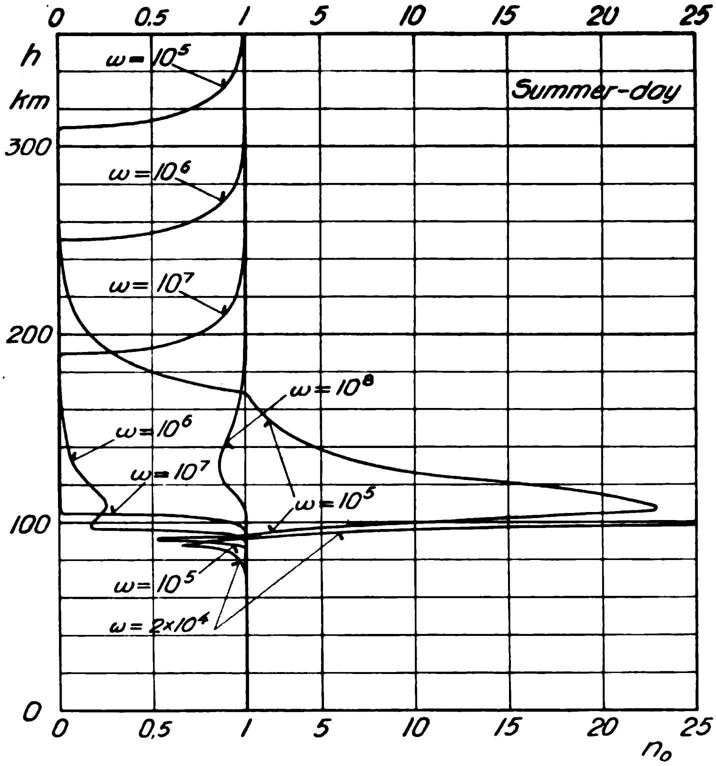

Fig. IX. 19. n as a function of h for $\omega = 2\cdot 10^4$, 10^5, 10^6, 10^7 and 10^8, summer day.

terrestrial magnetism requires a total conductivity of the atmosphere inf ull daylight condition equal to[3]

$$25\cdot 10^{-6}, \qquad \text{(e. m. u., cm)} \tag{5}$$

or about 8 times the preliminary value found by *Schuster*. However in these calculations no regard is paid to the influence on the conductivity of the magnetic field of the earth, and it appears from Fig. IX. 12 (p. 13 in the appendix). and from Chapt. VII formula (45f) that at high altitudes and for $\omega \to 0$ the conductivity in the direction of the magnetic field[4], — which is equal to the conductivity σ_0 without any magnetic field, — is very much greater than the conductivity at right angles to the magnetic field, *i. e.* $\sigma_0 \gg \sigma_{\mathrm{I}} = \sigma_{\mathrm{II}} = \sigma_{\perp}$ for $\omega \to 0$. We shall give some values of the ratio of these conductivities (from Fig. IX. 12):

[1] Chapt. V, p. 66.

[2] *S. Chapman:* Phil. Trans. (A). Vol. 213, p. 279—321, 1913; Vol. 214, p. 295—317, 1914; Vol. 215, p. 161—179, 1915; Vol. 218, p. 1—118, 1919; Vol. 225, p. 45—91, 1925.

[3] l. c., Vol. 218, p. 63.

[4] I. e. for waves with their electric vector in the direction of the magnetic field, see Chapt. VII, sect. 3, case α.

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$h =$	90	100	110	120	130	140	150	160	km
$\sigma_0 =$	$3.3 \cdot 10^{-17}$	$1.6 \cdot 10^{-14}$	$1.5 \cdot 10^{-12}$	$2.2 \cdot 10^{-11}$	$7 \cdot 10^{-11}$	$1 \cdot 10^{-10}$	$9 \cdot 10^{-11}$	$6.3 \cdot 10^{-11}$	e.m.u.
$\frac{\sigma_0}{\sigma_1} =$	1	3	32	300	1400	4000	6300	9000	

The total conductivity corresponding to σ_0 is

$$\Sigma \sigma_0 h \cong 450 \cdot 10^{-6} \qquad \text{(e. m. u., cm)} \quad (6)$$

and to $\sigma_\perp$:

$$\Sigma \sigma_\perp h \cong 0.22 \cdot 10^{-6}. \qquad \text{(e. m. u., cm)} \quad (7)$$

The intensity of the magnetic field used above is $H = 0.51$ gauss. If we take the value at the magnetical equator, namely $H = 0.35$ gauss[1] (compare also Fig. 21), we get (because $\sigma_{\perp(\omega=0)} \simeq \frac{e^2}{mc^2} \frac{v}{h^2}$, compare Fig. VII, 9):

$$\Sigma \sigma_\perp h \simeq 0.5 \cdot 10^{-6}. \qquad \text{(e. m. u., cm)} \quad (8)$$

The value (6) is 18 times greater and the value (8) 50 times smaller than *Chapman*'s value; this may probably be said to be fairly good agreement. But surely, in the further development of the theory of the solar and lunar diurnal variations of terrestrial magnetism it will be necessary to consider the influence of the earth's magnetic field on the conductivity in the upper atmosphere. Not until a theory improved in this respect has been worked out will it be possible to ascertain how nearly the conductivities required by the theory of radio wave propagation and by *Chapman*'s theory of diurnal magnetic variations agree with each other.

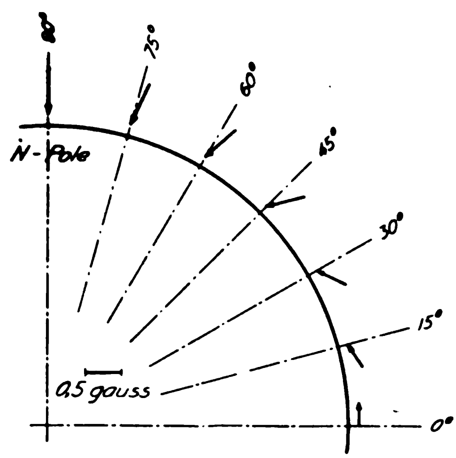

Fig. IX. 21. Approximate representation of the Earth's magnetic field. N is the magnetic North Pole.

[1] *G. Angenheister:* Handb. der Physik, Bd. XV, p. 274. 1927.

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CHAPTER X.

REFRACTION OF RADIO WAVES IN THE ATMOSPHERE.

1. *Refraction Caused by Varying Density and Humidity of the Atmosphere.*

If a plane electromagnetic wave, the front plane aa' of which at a certain moment passes through the centre O of the earth, is to continue to travel parallel to the curved surface MM' of the earth, then the wave front after the lapse of the time t must occupy the position bb' (see Fig. 1) where the plane bb' is similarly passing through the centre of the earth. We have then that the phase velocity v of the waves must vary with the height h in the following manner:

$$\frac{dv}{v} = \frac{dh}{R_0 + h} \quad \text{or} \quad \frac{dv}{dh} = \frac{v}{R_0 + h}. \tag{1}$$

where $R_0 = 6370$ km is the earth's radius.

We have

$$v = \frac{c}{n} \tag{2}$$

where $c = 3 \cdot 10^5$ km sec^{-1} is the velocity of light, and n is the refractive index of the medium.

According to *H. A. Lorentz* and *L. Lorenz*[1], the refractive index n of a gas depends on its specific density ϱ in the following manner:

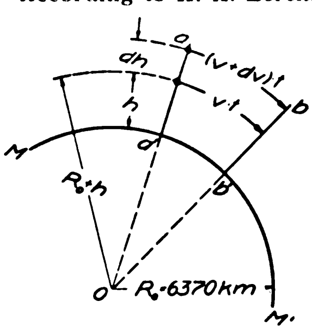

Fig. X. 1. The Condition necessary for Propagation of the Waves parallel to the Earth's Surface.

$$\frac{n^2 - 1}{n^2 + 2} \cdot \frac{1}{\varrho} = A' = \text{constant}. \tag{3}$$

If the value of the refractive index is very near to unity, (3) may be written approximately:

$$n - 1 = \varrho \frac{3}{2} A' = \varrho A, \tag{4}$$

where A is a constant.

The value of $(n-1) \cdot 10^6$ for red light and corresponding to a pressure of $p = 760$ mm and a temperature of 15° C. is given in Table 2 of Chapter IV for the principal gases contained in the atmosphere.

[1] *L. Lorenz:* Mém. de l'Acad. Roy. Copenhague, 5' Ser. Vol. 8, No. 5 (1869), Vol. 10. No. 8 (1875). Wied. Ann. Bd. 11, p. 70—103, 1880.
H. A. Lorentz: Wied. Ann. Bd. 9, p. 641—665, 1880.
K. Prytz: Wied. Ann. Bd. 11, p. 104—120, 1880.
St. Loria: Die Lichtbrechung in Gasen. (Braunschweig. 1914).

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Assuming, for the sake of simplicity, the temperature to be uniform throughout the height considered, we have according to Chapter IV

$$\varrho = \varrho_0 e^{-\frac{h}{H}}, \tag{5}$$

where H indicates the »height of the homogeneous atmosphere« (see Chapt. IV, 1) and ϱ_0 the specific density of the air for $h = 0$.

Inserting the expression (5) in the equations (2) and (1) we get:

$$R + h = \frac{H}{A\varrho_0} \cdot \left(1 + A\varrho_0 e^{-\frac{h}{H}}\right) e^{\frac{h}{H}}. \tag{6}$$

Thus, if the radius of the earth had this value R, then a gas of 760 mm pressure at the surface and with the constants A and H, would refract the electromagnetic waves so as to make the propagation at the altitude h parallel to the surface of the earth.

For $h = 0$ equation (6) assumes the form

$$R = \frac{H}{A\varrho_0}(1 + A\varrho_0) \tag{6a}$$

or, with sufficient approximation ($A\varrho_0 \ll 1$):

$$R = \frac{H}{A\varrho_0} = \frac{H}{n - 1}. \tag{6b}$$

This radius R corresponding to various gases is given in Table 2, Chapt. IV. The same table also gives the ratio between R and the earth's radius R_0. It will be seen that if the earth's radius had been 4.7 times as great as it is, then the electromagnetic waves, and rays of light, travelling horizontally at the earth's surface would continue to follow the latter, in such a manner that one would be able to »look around the earth«.

But it is also seen that this would only apply to the propagation in the very lowest portion of the atmosphere, since the value of R increases rapidly at greater heights, as it will appear from a comparison between the formulas (6) and (6a). The following remarks therefore apply only to propagation in the lower strata.

If the atmosphere consisted of carbon dioxide, then the radius of the earth would have to be only 1.9 times as great as it is in order to give the above mentioned case. For an atmosphere of krypton, for which $A\varrho_0 = 437 \cdot 10^{-6}$ and $H = 2.83$ km, we should find $R = 6630$ km, *i. e.* practically the same as the earth's radius.

In accordance with formula (6b) we may also express the same condition in the following manner: if the atmospheric pressure had been $4.7 \cdot 760 = 3572$ mm, then horizontal rays of light would follow the surface of the earth[1].

Although it has thus not been possible, solely on the basis of the refraction of the waves in the atmosphere, to explain the fact that radio waves appear in many cases to travel parallel to the earth's surface, nevertheless this refraction is of such magnitude that it is natural that some attempts have been

[1] This question has been treated for instance by: *J. A. Fleming:* Proc. Phys. Soc. Vol. 26, p. 318—332. 1914.

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made to explain radio wave propagation by other phenomena of a similar nature. Attention has here been directed especially towards the water vapour in the atmosphere. The question has been treated by *F. Kiebitz*[1] who assumed the constant A' in formula (3) to be of the same value for water vapour as for water. Writing accordingly $n^2 = (\varepsilon) = 81$ and $\varrho = 1$, we find $A' = 0.96$ or $A = \frac{3}{2} A' = 1.44$. For water vapour at 0^0 C. and $p = 760$ mm, corresponding to $\varrho = 8.04 \cdot 10^{-4}$, we have $\varrho A = 1158 \cdot 10^{-6}$. Taking the pressure of saturated water vapour at 20^0 C. to be 17.4 mm, its contribution to the value of ϱA will very nearly amount to

$$(\varrho A)_{\text{water vapour, } 20^0\text{C.}} = 1158 \cdot 10^{-6} \cdot \frac{17.4}{760} = 27 \cdot 10^{-6}.$$

At this temperature the increment of ϱA due to the water vapour would thus not amount to more than about 10 per cent.

F. Schwers[2] quite correctly points out that by this manner of reasoning too slight an influence is attributed to the water vapour. Experiments by *Baedeker*[3] have thus shown the constant A for the range of temperatures between 140^0 and 148.6^0 C. to be of a value 4.5 to 3.9 times as high as the one used by *Kiebitz*. *Schwers* then extrapolates from the values between 140^0 and 148.6^0 down to the value for 20^0 C. which he finds to be about 16 times as high as the one used above. This extrapolation, however, is very unreliable as also mentioned by *Schwers* himself. Using, nevertheless, his value of A we find

$$(\varrho \mathrm{A})_{\text{water vapour, } 20^0\text{ C.}} = 19300 \cdot 10^{-6} \cdot \frac{17.4}{760} = 443 \cdot 10^{-6}.$$

The total value for atmospheric air saturated with water vapour will then be

$$\varrho \mathrm{A} = (252 + 443) \cdot 10^{-6} = 695 \cdot 10^{-6}.$$

Even this value of ϱA gives too high a value of R according to the formula (6b), but nothing prevents the amount of water vapour from decreasing so rapidly upward that it might cause the path of radiation to bend with a radius equal to or smaller than the radius of the earth. This, however, can only be the case inside a very thin layer of air in the immediate vicinity of the earth's surface and at a relative high temperature of the air. This condition therefore evidently cannot form the basis for the long range ability of radio waves.

But in addition to this, *Schwers*' value of the constant A is undoubtedly too high. By measuring the increase in capacity produced by replacing the dry air in an air-condenser by air of the same temperature, but saturated with water vapour, we found the dielectric constant of air at 20^0 C. increased by not more than 0.5 $^0/_{00}$. Consequently for atmospheric air at 20^0 C. and saturated with water vapour the value of A cannot exceed $A = (252+250) \cdot 10^{-6} =$ abt. $500 \cdot 10^{-6}$.

To this comes further for long waves, *i. e.* for all radio waves, an eventual increase in refractive index due to the water present in the air in form of drops. According to *Humphreys*[4], the air contains at most 5.4 g of water in

[1] *F. Kiebitz:* Jahrb. d. drahtl. Tel. Bd. 7, p. 154—157. 1913.

[2] *F. Schwers:* Proc. Phys. Soc. Vol. 29, p. 150—157. 1916.

[3] *Baedeker:* Zeitschr. Phys. Chem. Vol. 36, p. 308. 1901.

[4] *W. J. Humphreys:* Physics of the Air, p. 268. (Philadelphia 1920). The said 5.4 g per cub. m correspond to 100 mm of rain per hour.

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the form of drops per cubic m. The corresponding refractive index n' may be computed from equation (3):

$$\frac{n'^2-1}{n'^2+2}=\frac{81-1}{81+2}\cdot 5.4\cdot 10^{-6}=5.3\cdot 10^{-6}$$

or $n'-1=8.1\cdot 10^{-6}$, since the constant A' for water ($\varrho=1$) in (3) is evidently equal to $\frac{81-1}{81+2}$.

Thus the presence of water in the form of drops in the air does not make any considerable contribution towards the increase in refractive index, and this quantity is reduced when the water contained in saturated air is condensed into drops.

Consequently, as indicated above, the value of $A\varrho=n-1$ will not exceed about $500\cdot 10^{-6}$, out of which about one half is due to the water vapour[1]. In the troposphere we may therefore expect to meet dividing surfaces where the value of n changes quite suddenly by an amount equalling about $250\cdot 10^{-6}$. If this applies to the wedge shown in Fig. 2, any waves of a direction of propagation perpendicular to the plane of symmetry of the wedge will be refracted through an angle $\alpha=0.14^0$. Furthermore, as shown in Chapter VIII, a reflection from such dividing surfaces may take place, when the angle of incidence is very nearly 90^0.

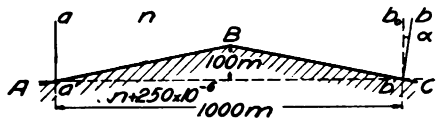

Fig. X. 2. The Refractive Index for the Wedge ABC is $n_1=n+250\cdot 10^{-6}$. With the dimensions given a wave passing the wedge will be refracted through the angle α determined by $\operatorname{tg}\alpha=0.0025$ or $\alpha=0.14^0$.

J. Guinchant[2] attempts to avoid the difficulties involved in explaining the curvature of the rays solely on the basis of the ordinary refractive conditions of the atmosphere, by assuming that in case of these relatively long waves the rays have a natural tendency to follow the surface of the earth. Under that assumption he is able to compute the refractive index n at the maximum height h which a ray with an earth angle ψ may reach, in the same manner as if the earth were plane. We have in that case

$$n_0 \cos\psi=n,$$

where $n_0=1.000300$ is the refractive index at the surface of the earth. The following table gives corresponding values of ψ and n, as well as the resulting values of the height h and the range D computed by *Guinchant*.

ψ =	10′	20′	30′	60′	80′
$(n-1)\cdot 10^6$ =	293	283	262	148	30
h =	0.130	0.550	1.26	6.24	17 km
D =	180	380	500	1670	5270 km.

It is difficult to see the justification of *Guinchant*'s assumption, but even granting this to be correct, *Guinchant'* theory does not explain in a satisfactory manner the features of propagation of the radio waves.

[1] *Delcelier, Guinchant* et *Hirsch* (L'Onde Électrique, 5, p. 189—216. 1926) arrive at the conclusion that for atmospheric air saturated with water vapour at 15^0 $n-1=400\cdot 10^{-6}$, which value is even slightly below the maximum value assumed by us.

[2] *J. Guinchant:* C. R. 179, p. 327—330. 1924.

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Although the variations considered in the refractive conditions of the atmosphere are thus insufficient to explain the fact that radio waves are largely able to follow the curvature of the earth, or even to return to the surface of the earth, these variations are nevertheless sufficient to produce appreciable deviations from that distribution of energy in the electromagnetic wave field which would exist in the atmosphere if the latter were homogeneous.

It appears clear from the above considerations that the variations of the refractive index are present only in the lower part of the atmosphere, the troposphere. At greater heights the air pressure and the water contents are too insignificant to play any appreciable part.

2. *Refraction Caused by Varying Index of Refraction of the Air due to Ions and Electrons.*

In Chapters VI and VII was shown that ionization may very considerably reduce the dielectric constant of the atmosphere and increase the conductivity and, thereby, modify the refractive properties of the same. We shall here consider merely the determination of the path of the ray, when the index of refraction of the atmosphere is given, and we assume, when nothing else is mentioned expressly, that n depends merely on the distance from the centre of the earth, in such a manner that the refractive index is of the same value throughout in any spherical surface concentric with the surface of the earth.

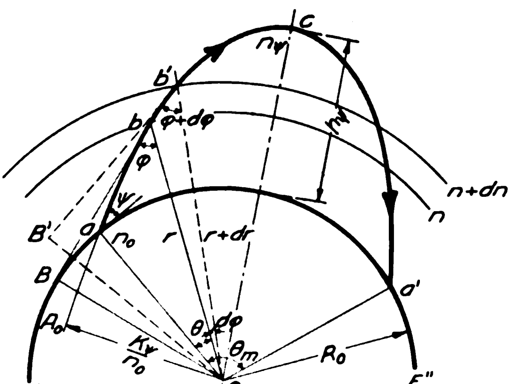

Fig. X. 3. Refraction in a Concentric Atmosphere.

In Fig. 3 $E'E''$ represents the surface of the earth, O its centre. A ray starts from the point a at an angle $\left(\frac{\pi}{2}-\psi\right)$ with the vertical at this point. The path $abb'ca'$ will in general be curved. We consider now the path at the two points b and b' where the refractive indices are n and $n+dn$, respectively. We further suppose the distance between b and b' to be so small that we may put $Ob' = Ob$. The distances OB and OB' from the centre of the earth to the tangents at points b and b' respectively, will then be determined by

$$\mathrm{OB} = \mathrm{Ob} \sin \varphi \text{ and } \mathrm{OB'} = \mathrm{Ob} \sin (\varphi + d\varphi). \tag{7}$$

According to the law of refraction we have:

$$n \sin \varphi = (n + dn) \sin (\varphi + d\varphi). \tag{8}$$

The equations (7) and (8) result in:

$$n \cdot \mathrm{OB} = (n + dn) \cdot \mathrm{OB'} = \text{constant} \doteq K_\psi = n_0 \mathrm{OA}_0 = n_0 R_0 \cos \psi, \tag{9}$$

where n_0 is the refractive index at the earth's surface[1].

[1] *J. M. Pernter:* Meteorologische Optik, p. 60 (Wien und Leipzig, 1902).
W. H. Eccles: The Electrician, Vol. 71, p. 969—970. 1913.

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Ordinarily the approximation $n_0 = 1$ will be sufficient, so that equation (9) becomes

$$n \cdot OB = nr \sin \varphi = K_\psi = R_0 \cos \psi, \tag{9'}$$

where $r = R_0 + h$ is radius vector to the point b.

A further consequence hereof is that the height h_ψ of the point c above the surface of the earth, where the direction of the ray is horizontal, will be determined by

$$n_\psi (R_0 + h_\psi) = K_\psi = R_0 \cos \psi \tag{10}$$

where n_ψ is the refractive index at the point c.

If the dependency of the refractive index on the height is known, h_ψ may at once be determined from equation (10). As mentioned below it may frequently be advantageous to make this determination by graphical means.

We shall next determine the shape of the path in a few definite cases. Indicating by θ the angle bOa, we have according to (9)

$$d\theta = \frac{K_\psi \cdot dr}{r\sqrt{n^2 r^2 - K_\psi^2}}. \tag{11}$$

Since n is assumed to be a function of r alone, the path may be determined by integration of (11), but only in rare cases may the integration be effected by means of known functions. *Eccles* notes a few cases, one of which will be mentioned here. He puts

$$n = \frac{K_\psi}{\cos \psi} \cdot \frac{R_0^s}{r^{s+1}} = \frac{R_0^{s+1}}{r^{s+1}}, \tag{12}$$

and the integration then gives

$$R_0^s \cos (s\theta - \psi) = r^s \cos \psi. \tag{13}$$

The angular range θ_m (see Fig. 3) is then determined by

$$\theta_m = \frac{2\psi}{s}, \tag{14}$$

and the maximum height of the ray above the earth is

$$h_\psi = R_0 \left(\frac{1}{\sqrt[s]{\cos \psi}} - 1 \right). \tag{15}$$

The maximum range will be attained for $\psi = \frac{\pi}{2}$ and is

$$\theta_{m\,max} = \frac{\pi}{s}, \tag{16}$$

but according to (15) the corresponding path will be infinitely long.

In the above we assumed $s > 0$. If on the contrary $s \leqq 0$, the ray will not return to the earth's surface. A few special cases should be mentioned here.

For $s = 0$ we find, instead of equation (13), by direct integration of equation (11):

$$\theta \operatorname{tg} \psi = \operatorname{logn} \frac{r}{R_0}. \tag{17}$$

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For $\psi = 0$ the ray will follow the surface of the earth. For $\psi > 0$ it will describe a spiral curve, and move farther and farther away from the earth (see Fig. 4).

For $s = -\frac{1}{2}$, equation (13) becomes

$$r = \frac{2R_0 \cos^2 \psi}{1 + \cos(\theta + 2\psi)}, \tag{18}$$

which represents a parabola having its focus at O and its vertex at a point with co-ordinates $r = R_0 \cos^2 \psi$ and $\theta = -2\psi$(see Fig. 5).

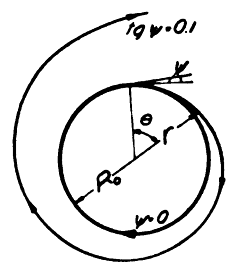

Fig. X. 4. Paths of Ray for $s = 0$, simultaneously with $\operatorname{tg} \psi = 0$ and $\operatorname{tg} \psi = 0.1$, respectively.

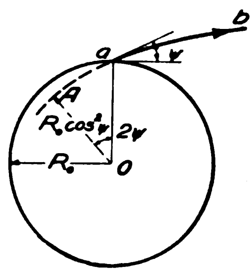

Fig. X. 5. The Parabolic Path Aab here shown corresponds to $n = \sqrt{\frac{R_0}{r}}$.

The variation of the refractive index, as determined by (12), is distinguished by its mathematical simplicity, and gives therefore a good synopsis of the influence of refraction on the shape of the ray, but it corresponds rather poorly to actual conditions, a fact also pointed out by *Eccles*.

In the following we shall mainly limit ourselves to approximate graphical methods for determining the path of the ray, and the integration of equation (11) will therefore not be further discussed. Instead we shall deduce a formula, found by *R. S. Heath*[1], for the curvature of the ray, a formula which will be used extensively in the graphical determination.

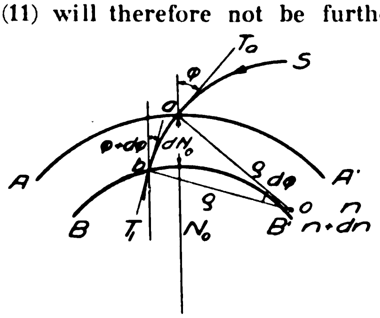

Fig. X. 6. Determination of the Radius of Curvature ρ of the Ray at the Point a. The centre of curvature is O.

In Fig. 6, AA' indicates a surface at which the refractive index has the constant value n. Correspondingly, the refractive index at the surface BB' situated infinitely near to AA' is equal to $n + dn$. The ray S strikes AA' at the point a where its tangent is T_a, while N_0 is the normal of the surface at the point a. The osculatory plane of the ray at the point a contains then T_a as well as N_0. Indicating by ρ the radius of curvature of the ray we have

[1] *A. Winkelmann:* Handb. d. Physik. VI, p. 491. 1906.
The corresponding formula given in *P. Drude:* Lehrbuch d. Optik, 2. Aufl., 1906, p. 292, is not correct.

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$$\varrho d\varphi = ab = -\frac{dN_0}{\cos\varphi}, \tag{19}$$

where dN_0 is the distance between the surfaces AA' and BB' at the point a.

According to the law of refraction we have

$$n \sin\varphi = (n + dn)\sin(\varphi + d\varphi)$$

and, consequently,

$$\frac{d\varphi}{dn} = -\frac{1}{n}\,\mathrm{tg}\,\varphi. \tag{20}$$

Equations (19) and (20) result in *Heath*'s formula:

$$\frac{1}{\varrho \sin\varphi} = \frac{1}{n}\cdot\frac{dn}{dN_0} = \frac{d\,\mathrm{lgn}\,n}{dN_0}. \tag{21}$$

It is easy to see that the locus for the centre of curvature of the ray, as the angle of incidence φ varies, is the straight line $b'bb''$ in Fig. 7.

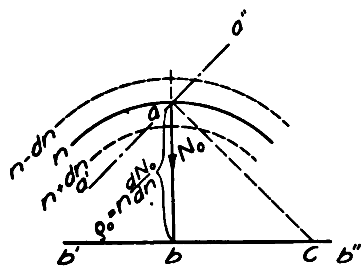

Fig. X. 7. c is the Centre of Curvature of a Ray passing through a, at which point the tangent of the ray is $a'a''$. The line $b'b''$ at right angle to the normal N_0 will then be the locus for the centres of curvature of all rays passing through the point a.

We shall next consider the distinguishing features of the path of the rays in cases where the variations of the refractive index are of the same character as those met with in the atmosphere. Where nothing else is expressly mentioned we assume, as above, that the refractive index is of the same value throughout in each horizontal surface, in other words that n depends merely on the altitude h. We make preliminarily the further assumption, in Fig. 8, that the earth's surface OO' is plane. Section I of Fig. 8 shows an example of n's dependency on h. The example selected presents no special features, but shows the simplest possible run of continuously varying values of n between the heights h' and h''. Below h' and above h'', n has its normal value, which we assume to be equal to unity. At the height h_2 n assumes its minimum value n_{min}, and at the heights h_1 and h_3 the (n, h)-curve has points of inflexion, so that the value of $\frac{dn}{dh}$ is minimum and maximum, respectively, at the above mentioned heights.

The path described by a ray, forming the angle φ with the vertical when it leaves the surface of the earth, will depend largely on the magnitude of φ. If φ is so great that $\sin\varphi > n_{min}$, then at the highest point of the path, where the tangent is horizontal, the ray will just be at the height h for which $n = \sin\varphi$. Subsequently the ray will bend downward, and at twice the distance from the starting point it will reach the earth once more. If we choose the angle $\varphi = \varphi_{min}$, where $\sin\varphi_{min} = n_{min}$, the corresponding ray (S_{max} in Fig. 8. V) will reach the height h_2, but it will only do so at an infinitely great distance from the starting point. If on the contrary $\sin\varphi < n_{min}$ and, consequently, $\varphi < \varphi_{min}$ then the ray, like S' in V, will penetrate the entire layer and continue its path outward into space, since after the height h'' has been passed, the direction will evidently be parallel to the original one. For $\varphi > \varphi_{min}$ the ray returns to the surface of the earth, for $\varphi < \varphi_{min}$ it continues its way into space. In the following we consider mainly the case where $\varphi \geqq \varphi_{min}$.

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To each individual value n' of the refractive index there corresponds thus one distinct value φ' of φ, viz. the value determined by the equation

$$\sin \varphi' = n', \tag{22}$$

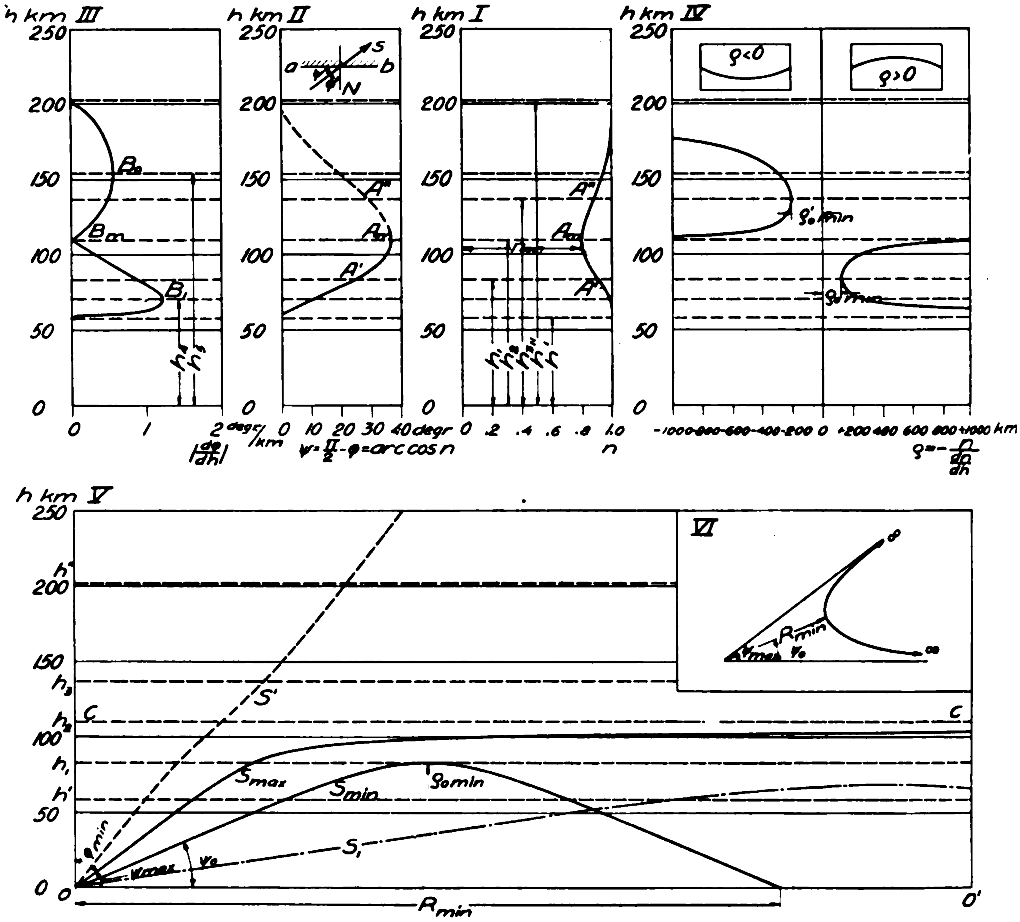

Fig. X. 8. Refraction of Rays in Plane Parallel Layers, the refractive indices of which are indicated in Section I.

which is such that a ray with the angle of incidence φ' will just reach the height h' where the refractive index is n'. Fig. 8. II shows, for $\psi = \frac{\pi}{2} - \varphi$, the ψ-curve corresponding to the n-curve in I. In Fig. 8 the equation (22) allows of two solutions, one corresponding to $h' < h_2$, the other one to $h' > h_2$. Evidently, only the former one of the two solutions can be used. The upper portion of the curve in II is therefore shown dotted.

If the intensity of the radiation is independent of φ, and if $\varphi > \varphi_{\min}$, then $\left|\frac{d\varphi}{dh}\right|$ will be a measure of the intensity of radiation at the apex of the path of the ray. The value of $\left|\frac{d\varphi}{dh}\right|$ is shown in III. At the points where $\left|\frac{d\varphi}{dh}\right| = 0$

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the intensity of the corresponding radiation will therefore be very small. $\frac{d\varphi}{dh}$ is maximum at the heights h_4 and h_5 where the ψ-curve has points of inflexion.

In order to form a clear idea of the path of the ray it is of great importance to find the radius of curvature at the various parts of the path. The radius of curvature ϱ_0 at the highest point of the ray is of special importance to the shape and range of the ray. According to (21) this radius is determined by

$$\varrho_0 = -n\frac{dh}{dn}, \tag{23}$$

where h, as usual, is taken positive in the upward direction.

Equation (21) shows that ϱ_0 is the minimum radius of curvature for a ray at the height considered. In Fig. 8. IV we have shown the variation of ϱ_0 as a function of h for the n-curve shown in I. At the heights h_1 and h_3 where the curvature of the n-curve changes sign, $|\varrho_0|$ is minimum, while $\varrho_0 = \infty$ at the heights where $\frac{dn}{dh} = 0$. A positive value of ϱ_0 corresponds to the concave side of the ray facing downward, while for $\varrho_0 < 0$ the concavity faces upward.

We consider next the dependency of the range on the angle of incidence, see Fig. 8. V. For $\varphi = \varphi_{min}$, corresponding to $\psi = \psi_{max}$, the ray S_{max} will reach the height h_2 for the minimum refractive index $n = n_{min}$, but it will only reach this height at an infinitely great distance from the starting point. Gradually, as φ increases, the range of the ray, from being infinitely great, will become smaller and smaller, until φ assumes the value corresponding to the height h_1, in which case the ray, S_{min}, will just reach up to this height, and the radius of curvature there, ϱ_{0min} (see Fig. 8. IV) will be the absolute minimum radius of curvature for any ray in the medium considered. The corresponding range is denoted R_{min} and, as it will appear from the following, this range, at any rate with some approximation, will be the minimum range attainable for the medium concerned, and for a returning ray starting from a point at the surface of the earth. If φ increases further, the apex of the path will be at lower altitude, such as S_1 in Fig. 8. V, but its range will be longer.

The shape of the (n, h)-curves to be considered will in many cases be of the character shown in Fig. 8. I, and in the following computations of the range we shall for the sake of simplicity replace the upper part of the actual path of the ray by a parabola with vertical axis and having at its apex a radius of curvature equal to ϱ_0, while the lower parts of the path are formed by rectilinear tangents to the parabola forming, at the points of intersection with the earth's surface, the angle φ with the vertical.

In order to ascertain how close this approximation will be and, at the same time, to clear up the question of the validity of the postulate that the minimum range will be attained by the ray having at its apex the absolutely minimum radius of curvature, we have shown, in Fig. 9, the path of the ray for a special case for which the path may be computed without difficulty, and where the (n, h)-curve at the same time is of a shape corresponding quite closely to the lower, »active«, portion of the (n, h)-curve in Fig. 8. We shall consider this case shown in Fig. 9 somewhat more closely.

The values of the refractive index are shown in I. From $h = 0$ up to $h = \frac{1}{2}b$ we have $n = 1$, but from $h = \frac{1}{2}b$ up to $h = \frac{3}{2}b$ we have

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$$n^2 = 1 - \alpha^2 + \alpha^2 \cos\left(\frac{\pi}{b}\,y\right), \tag{24}$$

where $y = h - \frac{1}{2}b$. Finally, for $h > \frac{3}{2}b$, we have $n \geqq \sqrt{1 - 2\alpha^2}$.

As shown in II, we displace the origin of co-ordinates to the point P at

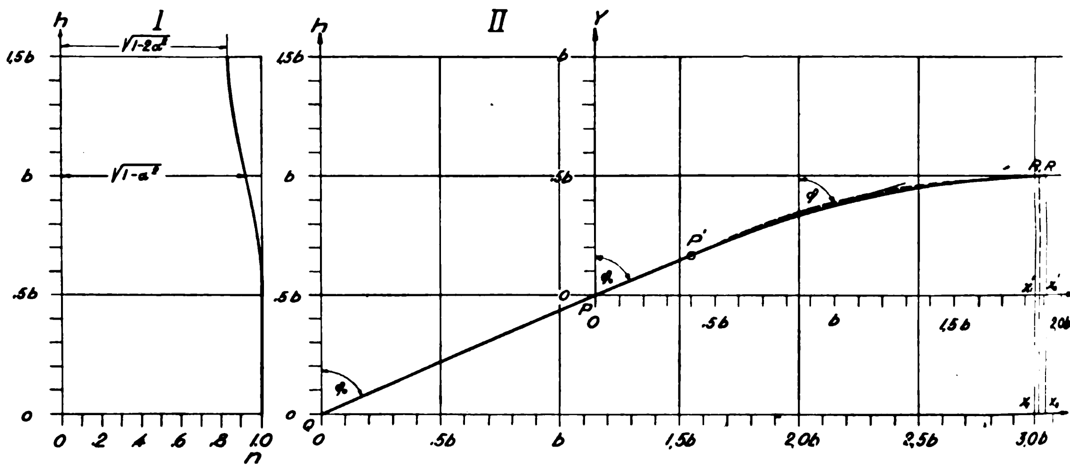
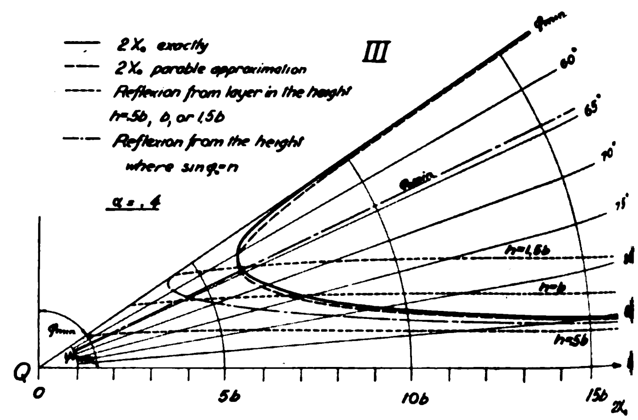


Fig. X. 9. I gives the shape of the n-Curve plotted according to equation (24). The Curve drawn in full in II shows the corresponding Path of the Ray, and finally the Full-Line Curve in III shows the Range. II and III also show the parabola-approximation, and III shows in addition some range-curves corresponding partly to reflection from the heights $h = 0.5\,b$, b and $1.5\,b$ and partly, for each individual direction of the ray, to reflection from the height for which $\sin\varphi = n$.



which the ray starting from Q strikes the plane $h = \frac{1}{2}b$. Denoting the abscissa by x' and the ordinate by y, the path of the ray will be determined by

$$x' = \sin\varphi_0 \int_0^y \frac{dy}{\sqrt{n^2 - \sin^2\varphi_0}} = \sin\varphi_0 \int_0^y \frac{dy}{\sqrt{\cos^2\varphi_0 - 2\alpha^2 \sin^2\left(\frac{\pi}{2b}\,y\right)}} \tag{25}$$

When we here put $y' = \frac{\pi y}{2b}$, equation (25) becomes:

$$x' = \frac{2b}{\pi}\,\mathrm{tg}\,\varphi_0 \int_0^{y'} \frac{dy'}{\sqrt{1 - \frac{2\alpha^2}{\cos^2\varphi_0}\sin^2 y'}}. \tag{26}$$

Since the parameter $\frac{2\alpha^2}{\cos^2\varphi_0} > 1$ in the cases where the ray returns to the

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surface of the earth — and only these cases are of interest to us — we insert in (26):

$$\sin^2 z = \frac{2\alpha^2}{\cos^2 \varphi_0} \sin^2 y',$$

whereby this equation becomes:

$$x' = \frac{\sqrt{2}b}{\alpha\pi} \sin \varphi_0 \int_0^z \frac{dz}{\sqrt{1 - \frac{\cos^2 \varphi_0}{2\alpha^2} \sin^2 z}}. \tag{27}$$

In order to determine the horizontal distance x_0' to the point R, for which the ray attains the maximum height and where, consequently, the tangent of the ray is horizontal, the limit of the integration has to be chosen in such a manner that at this point we have $dy = 0$ and, consequently, the denominator in the integral in equation (25) equal to zero. This condition results in

$$1 - \frac{2\alpha^2}{\cos^2 \varphi_0} \sin^2 y' = \cos^2 z = 0, \tag{28}$$

so that at this point we have $z = \frac{\pi}{2}$.

Consequently we have:

$$x_0' = \frac{\sqrt{2}b}{\alpha\pi} \sin \varphi_0 \int_0^{\frac{\pi}{2}} \frac{dz}{\sqrt{1 - \frac{\cos^2 \varphi_0}{2\alpha^2} \sin^2 z}} = \frac{\sqrt{2}b}{\alpha\pi} \sin \varphi_0 K\left(\frac{\cos \varphi_0}{\sqrt{2}\alpha}\right), \tag{29}$$

where K is the complete elliptic integral of the first kind with the modulus $\frac{\cos \varphi_0}{\sqrt{2} \cdot \alpha}$.

The horizontal distance x_0 from the starting point Q to the apex is thus

$$x_0 = \frac{b}{2} \operatorname{tg} \varphi_0 + x_0' = \frac{b}{2} \operatorname{tg} \varphi_0 + \frac{\sqrt{2}b}{\alpha\pi} \sin \varphi_0 K\left(\frac{\cos \varphi_0}{\sqrt{2}\alpha}\right) \tag{30}$$

and the range of the ray is $2x_0$.

The range for various values of φ_0 is shown in III which also shows the direction of radiation, marked $\varphi_{0_{min}}$, corresponding to a path of ray having, at its apex, just the minimum radius of curvature $\rho_{0_{min}}$. The corresponding range is seen to be very nearly equal to the minimum value.

In II there is shown, by a dotted line, the above mentioned parabolic approximation. The horizontal distance x_1 of the apex from Q is here slightly smaller than for the actual ray path, but the difference is only slight. In III a dotted curve shows the ranges corresponding to the parabolic approximation. It is seen that the two range-curves may be considered to be co-incident with an approximation sufficient for the object in view.

For use in a following chapter we have further shown, in III, the ranges corresponding to the case that the rays emitted from Q are reflected from plane layers of heights $0.5b$, b and $1.5b$, respectively, above the surface of the earth. Finally the dot-and-dashed line shows the range in case the ray is reflected from a plane, horizontal layer at the height for which $n = \sin \varphi_0$, *i. e.* in each individual case from the maximum height to which a ray with an angle of incidence φ_0 is able to penetrate. It will be seen that none of the four last mentioned range-curves gives any useful approximation to the true one.

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There is in fact an important and fundamental difference between the true range-curves and the range-curves corresponding to reflection at some definite height. While in the latter case one, and only one, definite angle of incidence corresponds to each individual range, we have, under actual conditions, always the same range for two different angles of incidence, one of which is greater and the other smaller than the one corresponding to R_{min}. This question will now be considered somewhat more closely.

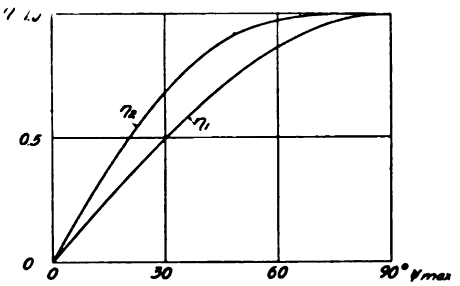

Fig. X. 10. The Ratio between the Reverting and the Total Radiated Effect. η_1 corresponds to a transmitter radiating uniformly in all directions, η_2 to a vertical dipole.

It is evident that for a (n, h)-curve of the shape shown in Fig. 8. I – and this variation in the refractive index is the simplest onè to be imagined — there will always exist a minimum range R_{min} for the refracted rays, in the sense that rays emitted from Q cannot revert to the earth's surface at any shorter distance from Q than R_{min}.

We shall next quite briefly consider the main features of the distribution of energy. Under the above mentioned assumption that the intensity of the radiation is independent of the angle $\psi = \frac{\pi}{2} - \varphi$, that fraction η_1 of the radiation energy which reverts to the earth's surface will be determined by

$$\eta_1 = \sin \psi_{max}. \tag{31}$$

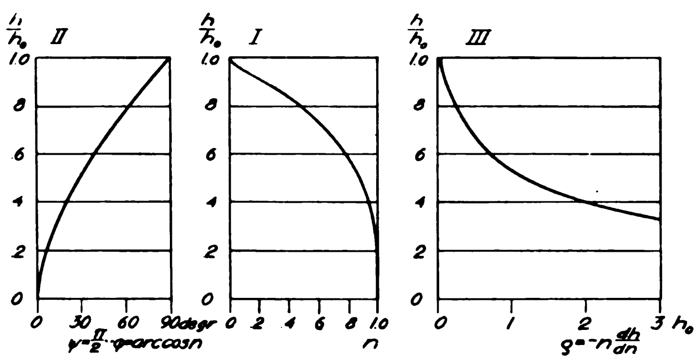

Fig. X. 11. Diagrams Indicating the Shape of the n- and ϱ-Curves when the Refractive Index decreases to Zero.

If the source of radiation is a dipole with an intensity of radiation proportional to $\cos^2 \psi$, we have

$$\eta_2 = \tfrac{9}{8} \sin \psi_{max} + \tfrac{1}{8} \sin (3\psi_{max}). \tag{32}$$

Fig. 10 shows the variations of η_1 and η_2 as functions of ψ_{max}.

It appears further from an inspection of Fig. 8 and Fig. 9 III that the intensity of the radiation at the surface of the earth will be maximum at the distance R_{min}, and will then decrease as the distance increases. The shape of the (n, h)-curve will of course influence the energy distribution within the range

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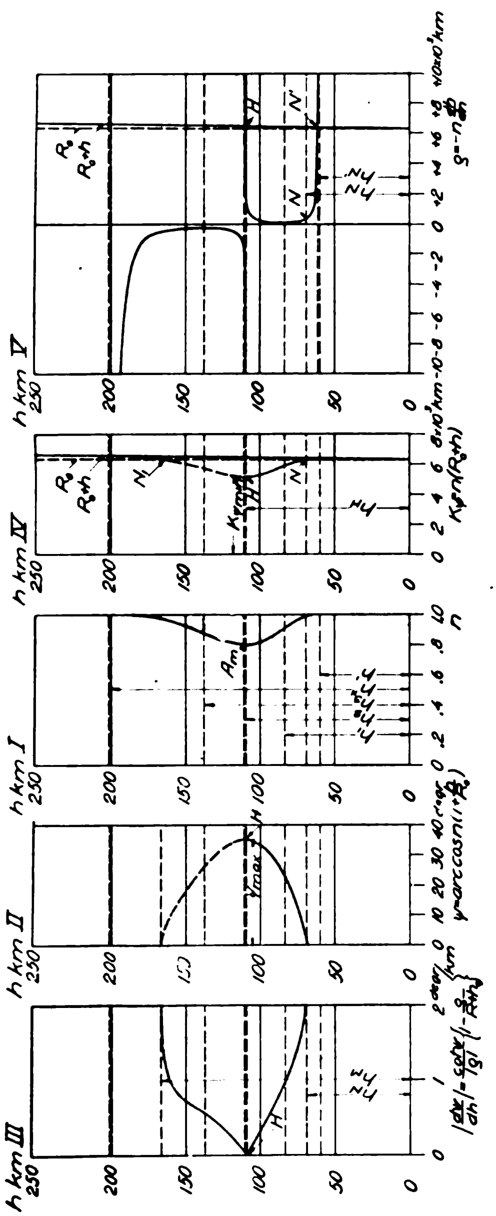
interval $R_{min} \rightarrow \infty$, but for a continuously varying (n, h)-curve of the shape shown these general remarks will always apply.

Besides the attenuation which is due to dispersion of the energy of radiation,



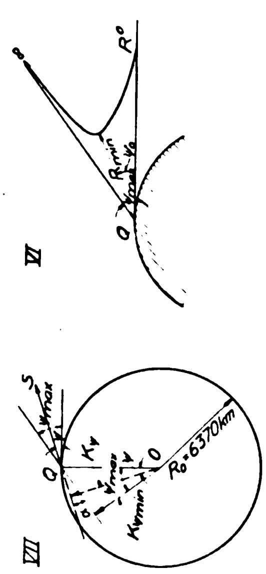


Fig. X. 12. Treatment of the Radiation Problem, taking into consideration the Curvature of the Earth.

and which is the only attenuation to which we have paid any attention hitherto the rays will also be attenuated on account of the conductivity of the ionized layers. We shall later come back to this question.

In Fig. 11 we have shown, diagrammatically, another shape of the (n, h)-curve for which n is equal to zero for $h = h_0$. Here $\varphi_{min} = 0$, and, consequently, $\psi_{max} = \frac{\pi}{2}$, so that even a vertically incident ray will be reflected in this case. This corresponds to $\varrho = 0$ for $n = 0$. A surface for which $n = 0$ will thus be perfectly impervious to the radiation.

We shall now proceed to consider the case where the arth is assumed to be a sphere of radius $R_0 = 6370$ km. We assume that the value of n as a function of h is given by the curve shown in Fig. 12 I. In section IV of the same figure the fully drawn straight line represents the value of $R_0 + h$, while the curved line shows the value of $n(R_0 + h)$. A ray S leaving the surface of the earth under an angle ψ (see Sec-

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tion VII of Fig. 12) will then according to equation (10) reach an altitude h_ψ determined by

$$n\,(R_0 + h_\psi) = K_\psi = R_0 \cos \psi, \tag{10}$$

where K_ψ is the distance from the centre of the earth to the ray S at the starting point.

In the manner here indicated we are able to construct the (ψ, h)-curve (see Section II) corresponding to the n-curve in I, since, according to (10), we have

$$\psi = \operatorname{arc\,cos} \left[n \left(1 + \frac{h}{R_0} \right) \right]. \tag{33}$$

On the basis of the (ψ, h)-curve we may then construct the $\left|\frac{d\psi}{dh}\right|$-curve shown in III, since, according to (33) and (10):

$$\left|\frac{d\psi}{dh}\right| = \left(\frac{1}{\varrho_0} - \frac{1}{R_0 + h} \right) \cdot \cot \psi, \tag{34}$$

where

$$\varrho_0 = - \frac{n}{\left(\frac{dn}{dh}\right)} \tag{35}$$

is the radius of curvature of horizontal rays at the height h [1] (see equation (23) above).

The value of ϱ_0 is plotted in Fig. 12. V which also shows the value of $R_0 + h$. This latter straight line intersects the ϱ_0-curve at the points N' and H situated at the heights $h_{N'}$ and h_H. At the last mentioned height, where $\varrho_0 = R_0 + h_H$, the $n(R_0 + h)$-curve must have a vertical tangent, since rays reaching further up into the atmosphere will never come back to the surface of the earth. These rays, consequently, cannot have any highest point. K_ψ must therefore at this point H assume its minimum value $K_{\psi min}$ corresponding to a maximum value of ψ as determined by

$$\cos \psi_{max} = \frac{K_{\psi min}}{R_0}. \tag{36}$$

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[1] *R. Gans* (Ann. d. Physik. (IV). Bd. 47, p. 709—736. 1915) has demonstrated that certain complications will occur at this very point, where the ray according to the rules of geometrical optics runs parallel to the surface $n =$ constant, since the actual ray path presents an angular bend indicated in the attached Fig. a by a fully drawn line, while the dotted geometrical path, which we have solely considered, has a continuous run. Since the ray path only in the immediate vicinity of the critical surface deviates from the path determined in accordance with the rules of geometrical optics, we have used these simple rules here exclusively in determining the shape of the path. Later on, in discussing the attenuation of the ray along a curved path, it will be necessary to return to this question.

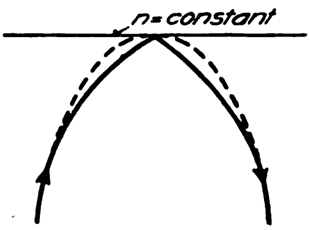

Fig. X. a. The Actual and the »Geometrical« Ray according to *Gans*. The geometrical ray is shown dotted.

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That the tangent at the point H of the $n(R_0 + h)$-curve is actually vertical may also be easily seen in the following manner. At the point considered we have

$$\frac{d[n(R_0 + h)]}{dh} = (R_0 + h)\frac{dn}{dh} + n, \tag{37}$$

but according to equation (35) the right-hand side of this equation is equal to zero, since $\varrho_0 = R_0 + h$ according to the assumption made.

The altitude h_N of the lower of the two points N and N_1 for which $R_0 = n(R_0 + h)$, *i. e.* the maximum height to which the rays $\psi = 0$ may penetrate, must exceed the height $h_{N'}$ to the lower point of intersection N' between the ϱ_0-curve and the line $(R_0 + h)$ (see Fig. 12 V). This is also easily demonstrated in the following manner: At the point N we have

$$n_N = \frac{R_0}{R_0 + h}. \tag{38}$$

Since now $n = 1$ up to $h = h'$, and thereafter decreases more and more rapidly within the field considered, we have:

$$\left|\frac{dn}{dh}\right| > \frac{R_0}{(R_0 + h)^2} \tag{39}$$

because, if n decreased uniformly from unity at the height zero to the value n_N at the height h_N according to the formula $n = \frac{R_0}{R_0 + h}$, then we should have

$$\left|\frac{dn}{dh}\right| = \frac{R_0}{(R_0 + h)^2}.$$

Equations (38) and (39) result in:

$$\varrho_0(h_N) = \left[\frac{n}{\left(\frac{dn}{dh}\right)}\right]_{h=h_N} < n_N \cdot \frac{(R_0 + h)^2}{R_0} = R_0 + h. \tag{40}$$

which was to be demonstrated.

Also in this case there will be a certain minimum range R_{min} corresponding approximately to the range of rays having their highest point at the height h_1 and therefore at their apex the absolute minimum radius of curvature ϱ_{0min} which any ray can have, under the conditions given. The corresponding value of ψ is denoted by ψ_0 (see Section VI, which shows the dependency of the range on ψ). For $\psi = 0$ the range has a finite value R^0. For $\psi = \psi_{max}$ the range is infinite. But as in the case of a plane surface of the earth, the energy will be a maximum in the vicinity of R_{min}, and then decrease gradually as the distance from this point increases. It will, however, increase again and reach a maximum at R^0, and then again decrease as the distance increases[1].

---

[1] After the above representation of the refraction of the rays in the atmosphere had been written, several authors published similar treatments of the same subject. We mention *e. g.*: *W. G. Baker* and *C. W. Rice:* »Refraction of Short Radio Waves in the Upper Atmosphere« (A. J. E. E. 1926) and *H. Lassen:* Jahrb. d. drahtl. Tel. Bd. 28, p. 139 – 147 (Nov. 1926). Since the present presentation may perhaps offer certain advantages at some points, we have thought it proper to give it here in its entirety.

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As mentioned above under the treatment of the propagation of waves along a plane surface, we may also in this case often use a parabola-approximation for the calculation of the range b of a ray. As shown in Fig. 13 we replace the top portion CED of the path of the ray by a parabola with the apex E

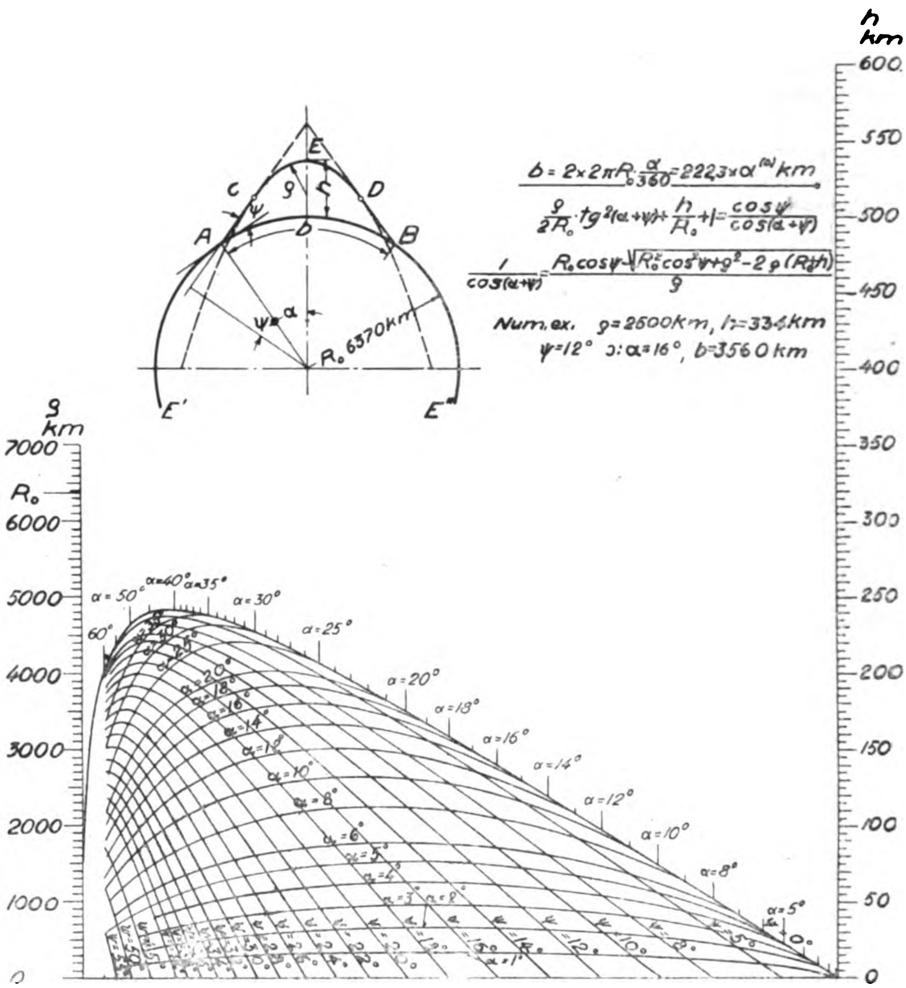

Fig. X. 13. An Abac for the Ranges of Electric Waves, forming at the starting point an angle ψ with the earth's surface and reaching the maximum height h, with a radius of curvature ϱ, the path being replaced by a parabola CED with a radius of curvature ϱ at the apex E and by the tangents AC and DB forming the angle ψ with the earth's surface at A and B. A straight line connecting, on the abac, the given values of ϱ and h intersects the curve marked ψ at a point corresponding to the unknown value of the angle α, whereafter the range is $b = 2 \cdot 2\pi R_0 \cdot \frac{\alpha}{360} = 222.3 \cdot \alpha^0$ km.

situated at the maximum height h reached by the ray concerned, while the radius of curvature of the parabola at this point is equal to the radius of curvature ϱ of a horizontal ray at the height h. The lower parts of the path of the ray consist of the straight lines AC and BD, tangents to the parabola

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and forming the angle ψ with the horizontal plane at the starting point A and the point of arrival B. A determination of the range b may be effected conveniently by means of the abac shown in Fig. 13, if the values of ϱ, h and ψ are known. If the highest point of the ray is situated at the altitude for which the radius of curvature of the horizontal rays has its minimum value ϱ_0, then the range will be approximately equal to the value R_{min}.

This parabola-approximation fails altogether in the important case of »long distance rays« where the radius of curvature at the top is very nearly equal to $(R_0 + h)$.

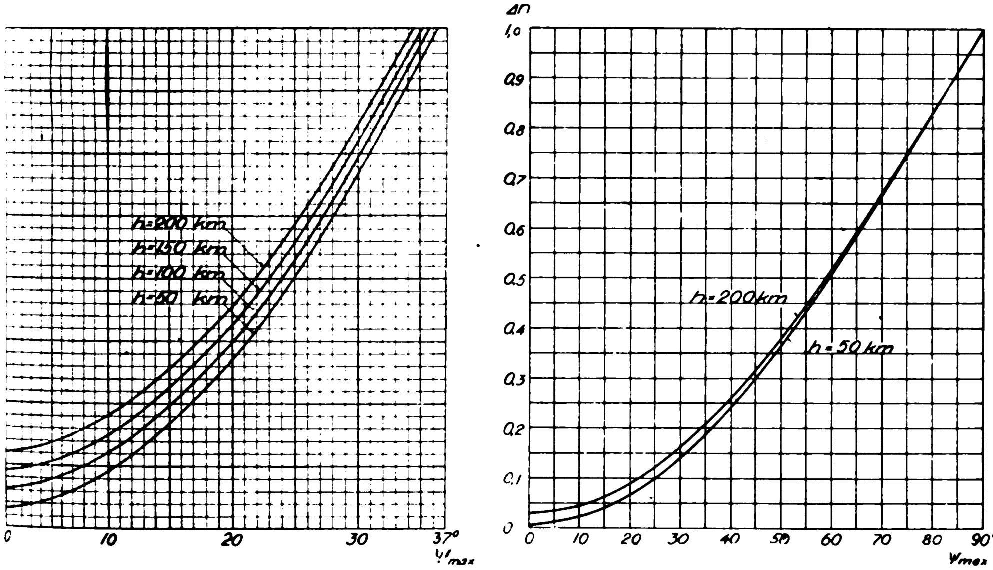

Fig. X. 14 a. Fig. X. 14 b.

The Minimum Value of Δn for Reverting Rays, as a Function of ψ_{max} and h.

It is very easy to determine the minimum value of Δn for which the rays return to the surface of the earth, when the angle ψ_{max} is given, since in that case we have, according to (10), the simple relation

$$n(R_0 + h) = K_{min} = R_0 \cos \psi_{max}. \tag{41}$$

Inserting here $n = 1 - \Delta n$, we find:

$$\Delta n = 1 - \frac{\cos \psi_{max}}{1 + \frac{h}{R_0}}, \tag{42}$$

an equation determining the minimum decrease in the refractive index for which the ray ψ_{max} will again be bent back towards the surface of the earth. Fig. 14 a and b show Δn as a function of h and ψ_{max}.

The ratio η between the reverting and the total radiated energy is also in this case given by equations (31) and (32) and is represented in Fig. 10.

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3. *Determination of the Time Required for a Signal to cover the curved path of a ray. (Phase-velocity and group- or signal-velocity).*

It has been attempted to determine experimentally the height to which the waves penetrate into the atmosphere by measuring the difference in time between the arrival at the receiver at B of an impulse transmitted from the sender at A directly to B, and the arrival at B of the same impulse having traversed the path $AacbB$ in Fig. 15. If we assume, preliminarily, that the curved portion of the path acb is replaced by the rectilinear pieces ac' and bc', corresponding to the ray being reflected from a layer at the height H', and if we make the further assumption that all the rectilinear paths are

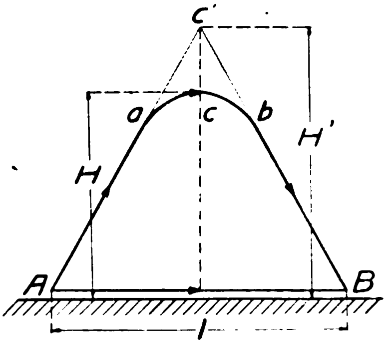

Fig. X. 15. AB is the Direct Path, $AacbB$ is the Path of the Ray Reverting from the Upper Atmosphere ($n \leqq 1$).

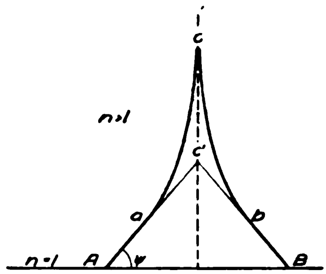

Fig. X. 16. This Figure is Analogous to Fig. 15, but here $n \geqq 1$.

traversed at the velocity $c = 3 \cdot 10^{10}\ \mathrm{cm\ sec}^{-1}$, then the time interval τ between the arrivals at B will be determined by:

$$\tau = \frac{Ac' + c'B - AB}{c} = \frac{\sqrt{l^2 + 4H'^2} - l}{c}, \tag{43}$$

which equation may serve to determine H' when l and τ are known.

If the refractive index at higher altitudes is greater than unity, then the ray will be bent upward, as shown by Aac in Fig. 16. Total reflection will only occur at the height c for which $n = \infty$, *i. e.* it will never occur. A partial reflection will occur at lower height, and the reflected ray will strike the earth once more at B. This case, however, is of no particular interest in the present connection.

In order that this determination according to equation (43) may be fairly reliable it is evidently necessary that l be not too great in comparison with H', and preferably considerably smaller than this height. Since the time τ is so short, for $H' = 150$ km only about 0.001 second, a measurement of τ with the accuracy required would involve serious difficulties. Besides, the wave-front of the train of waves returning from the upper atmosphere will generally be different from the wave front of the train of waves passing directly from A to B, so that the impulse will not build up in exactly the same manner in the two cases. Finally the path of the reverting ray will not be $Ac'B$, but $AacbB$, where the curved portion acb of the path is traversed by the impulse at a velocity which differs from the velocity c of light in free

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space, and which also differs from the phase-velocity corresponding to the refractive index. As shown in Chapter VIII the phase-velocity is determined by

$$v = \frac{c}{n},$$

where n as usual is the refractive index of the medium for the frequency considered. The phase-velocity is the only one we have considered until now, and it is the velocity determining the path of the ray when the wave train has been maintained so long that the state prevailing along the path of the ray may be considered to be stationary. But the phase-velocity is not the one determining the time required for an impulse to pass the path acb. Here the group- or signal-velocity u is the deciding factor, and this is determined by[1]:

[1] Lord *Rayleigh*: 1) Theory of Sound. Vol. I, § 191. 1877. 2) Proc. London Math. Soc. IX, p. 21—26, 1877. (Scientific Papers. Vol. I, p. 322—330).

Osborne Reynolds: »Nature«, Aug. 23, 1877. (Scientific Papers, Vol. I, p. 198—203).

The formula (44) may be deduced in the following manner: Two waves a and a_1, see Fig. 17, both of amplitude 1, are moving in the same direction. Their periods

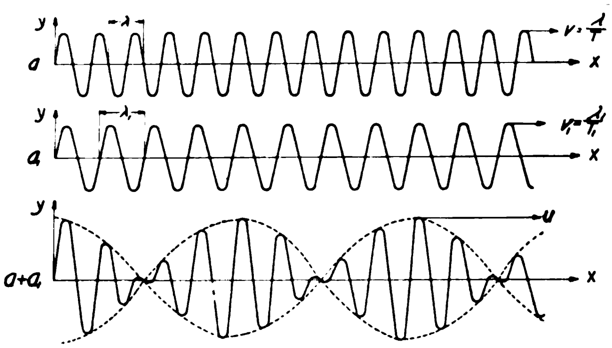

Fig. X. 17. The Waves a and a_1 have Phase-Velocities v and v_1. The Resultant Wave, $a + a_1$, has the Group-Velocity u.

T and T_1, wave lengths λ and λ_1 and phase-velocities v and v_1 are slightly different. The resultant wave $a + a_1$ will then be

$$y = 2\cos\pi\left[t\left(\frac{1}{T} - \frac{1}{T_1}\right) - x\left(\frac{1}{\lambda} - \frac{1}{\lambda_1}\right)\right]\cdot\sin\pi\left[t\left(\frac{1}{T} + \frac{1}{T_1}\right) - x\left(\frac{1}{\lambda} + \frac{1}{\lambda_1}\right)\right]. \qquad \text{(a)}$$

At the time $t = 0$ the resultant wave is maximum at the point of origin $x = 0$. This maximum will be moving, and the motion will be determined by the following equation:

$$t\left(\frac{1}{T} - \frac{1}{T_1}\right) - x\left(\frac{1}{\lambda} - \frac{1}{\lambda_1}\right) = 0. \qquad \text{(b)}$$

The maximum amplitude will consequently move at a velocity u_1 determined by

$$u_1 = \frac{x}{t} = \frac{\dfrac{1}{T} - \dfrac{1}{T_1}}{\dfrac{1}{\lambda} - \dfrac{1}{\lambda_1}} = \frac{\dfrac{v}{\lambda} - \dfrac{v_1}{\lambda_1}}{\dfrac{1}{\lambda} - \dfrac{1}{\lambda_1}}. \qquad \text{(c)}$$

For $T_1 \rightarrow T$, $\lambda_1 \rightarrow \lambda$ and $v_1 \rightarrow v$, the formula (c) becomes equivalent to (44). The thus determined velocity u is the group- or signal-velocity. (u is the velocity of propagation of an impulse carried by a carrierwave of a wave length λ).

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$$u = \frac{d\left(\frac{v}{\lambda}\right)}{d\left(\frac{1}{\lambda}\right)} = v - \lambda\,\frac{dv}{d\lambda}. \tag{44}$$

Inserting here $\lambda = \frac{2\pi}{\omega}\,v$ and, consequently,

$$d\lambda = 2\pi\,\frac{\omega\,dv - v\,d\omega}{\omega^2}$$

we find[1]:

$$\frac{u}{v} = \frac{1}{1 - \frac{\omega}{v}\,\frac{dv}{d\omega}}. \tag{45}$$

Substituting $v = \frac{c}{n}$, where n as usual is the refractive index of the medium and $c = 3\cdot 10^{10}$ cm sec^{-1}, equation (45) becomes:

$$u = \frac{v}{1 + \frac{\omega}{n}\cdot\frac{dn}{d\omega}} = \frac{c}{n + \omega\,\frac{dn}{d\omega}}, \tag{46}$$

which may serve to determine the group- or signal-velocity, when the refractive index is known as function of ω. For n independent of ω we have $u = v$. For a medium without dispersion the group-velocity and the phase-velocity are consequently alike.

Depending on the circumstances the group-velocity may be greater or smaller than the phase-velocity[2], but in the case treated here it must necessarily

---

[1] *G. Breit* and *M. A. Tuve:* Phys. Rev. (II), Vol. 28, p. 554—575. 1926.

[2] According to *Reynolds* and *Rayleigh* the group-velocity is also determined by:

$$u = \frac{\text{Amount of Energy passing a Cross-Section per Unit Time}}{\text{Amount of Energy per Unit Length for the same Cross-Section}}$$

For the sake of orientation the following synopsis, which is due mainly to Lord *Rayleigh*, may perhaps be of interest:

Phase-Velocity $v =$	*Refractive Index* $n =$	*Group-Velocity* $u =$	*Remarks*
$k\lambda^{s} = k_0\omega^{\frac{s}{s-1}}$	$K\omega^{\frac{s}{1-s}}$	$(1 - s)\,v$	General Formulas
$k_1\lambda = k^{I}\omega^{\pm\infty}$	$K_1\omega^{\mp\infty}$	0	Disconnected Pendulums[3] Waves in a Corn-Field
$k_2\lambda^{\frac{1}{2}} = k^{II}\omega^{-1}$	$K_2\omega$	$\frac{1}{2}\,v$	Water-Waves in Deep Sea
$k_3 = k^{III}$	K_3	v	Waves in a Medium Without Dispersion
$k_4\lambda^{-\frac{1}{2}} = k^{IV}\omega^{\frac{1}{3}}$	$K_4\omega^{-\frac{1}{3}}$	$\frac{3}{2}\,v$	Capillary Waves on a Water Surface
$k_5\lambda^{-1} = k^{V}\omega^{\frac{1}{2}}$	$K_5\omega^{-\frac{1}{2}}$	$2v$	Transverse Wave Along a Rod
$k_6\lambda^{-\infty} = k^{VI}\omega$	$K_6\omega^{-1}$	∞	Limiting Case

[3] Lord *Rayleigh*: Phil. Mag. (5). Vol. 46, p. 567. 1898.

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be smaller than or at most equal to c. According to the equation (7) in Chapter VIII we have:

$$n = \sqrt{\sqrt{\frac{\varepsilon^2}{4} + \left(2\pi c^2 \frac{\sigma}{\omega}\right)^2} + \frac{\varepsilon}{2}} \tag{47}$$

where ε and σ, according to formulas (36) and (21) in Chapter VI are determined by

$$\varepsilon = 1 - N 4\pi \frac{e^2}{m} \frac{1}{\nu^2 + \omega^2} \quad \text{and} \quad \sigma = N \frac{e^2}{mc^2} \cdot \frac{\nu}{\nu^2 + \omega^2}, \tag{48}$$

N indicating the number of electrons (or ions) per c c, the earth's magnetic field left out of consideration. When these expressions are inserted into (46) this formula may be shown to give $u \leqq c$. In several cases it may be of interest to be able to make an estimate of the time required for a signal to traverse the curved portion of the path. This applies for instance to the attempts mentioned below to determine the height of the ionized layers by a measurement of time. We consider therefore Fig. 18, where *ab* is the original direction of the ray, while *cd* is a portion of the curved path. In a layer of thickness *dh*, the element of length *ds* of the curved path is determined by

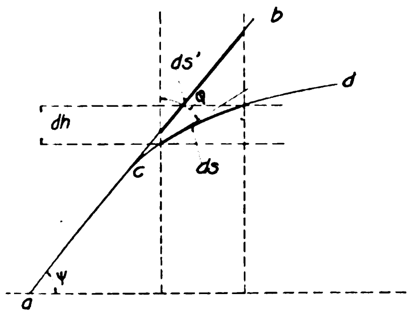

Fig. X. 18. The Original Direction of the Ray *ab* forms the angle ψ with the horizontal layer. From *a* to *c* the refractive index *n* is equal to unity, above *c* *n* is smaller than unity. *cd* is the curved ray.

$$ds = \frac{dh}{\cos\varphi}.$$

The element *ds'* of the straight line *ab* having the same horizontal projection as *ds* is determined by

$$ds' = \frac{\operatorname{tg}\varphi}{\cos\psi}\, dh.$$

For the ray path the equation $n \sin\varphi = \cos\psi$ is valid and, consequently:

$$n = \frac{\cos\psi}{\sin\varphi}.$$

We have heretofore assumed $n \leqq 1$, but exactly the same relations hold good in case $n > 1$, the only difference being that the path, as shown in Fig. 19, will be situated above the original direction of the ray *Ac*.

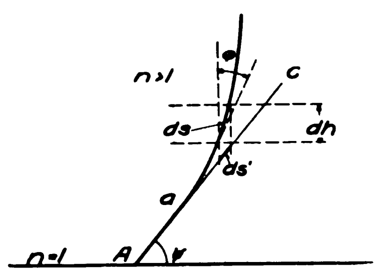

Fig. X. 19. This Figure is analogous to 18, but applies only in case of $n > 1$.

The phase velocity is $v = \frac{c}{n}$, and according to (46) the time required for a signal to traverse the path element *ds* is determined by:

$$dt = \frac{1}{u} \cdot \frac{dh}{\cos\varphi} = \frac{\cos\psi}{\sin\varphi \cos\varphi} \left(1 + \frac{\omega}{n} \frac{dn}{d\omega}\right) \cdot \frac{dh}{c}. \tag{49}$$

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The time dt' required to traverse the path element ds' at the velocity c is

$$dt' = \frac{ds'}{c} = \frac{\operatorname{tg}\varphi}{\cos\psi} \cdot \frac{dh}{c}. \tag{50}$$

The ratio between the two time intervals is

$$\frac{dt}{dt'} = \frac{\cos^2\psi}{\sin^2\varphi}\left(1+\frac{\omega}{n}\frac{dn}{d\omega}\right) = n^2\left(1+\frac{\omega}{n}\frac{dn}{d\omega}\right) = n^2 + \tfrac{1}{2}\omega\frac{d(n^2)}{d\omega}. \tag{51}$$

Assuming now that in formula (47) σ may be disregarded, which will frequently be the case, then

$$n^2 = \varepsilon = 1 - N4\pi\frac{e^2}{m}\frac{1}{\nu^2+\omega^2}. \tag{52}$$

and when this value is inserted in (51), we find:

$$\left(\frac{dt}{dt'}\right)_{\sigma=0} = 1 - N4\pi\frac{e^2}{m}\frac{\nu^2}{(\nu^2+\omega^2)^2}. \tag{53}$$

For $\nu=0$, $dt=dt'$, that is to say the same time is required for a signal to traverse the curved path acb in Fig. 15 as would be required to traverse the two rectilinear portions ac' and $c'b$ at the velocity c. This special case, where $\nu=0$, has already been treated by *Breit* and *Tuve* in their above mentioned paper.

Formula (53) shows that generally the curved path acb (Fig. 15) will be traversed by a signal in shorter time than the way $ac'+c'b$ is traversed at the velocity c.

According to the assumption made, viz. that $\sigma=0$, the waves cannot enter into regions for which $\varepsilon<0$. We must therefore according to (52) have

$$N4\pi\frac{e^2}{m}\frac{1}{\nu^2+\omega^2} \leqq 1.$$

Inserting this relation in (53) we find:

$$\frac{dt}{dt'} \geqq 1 - \frac{\nu^2}{\nu^2+\omega^2} = \frac{\omega^2}{\nu^2+\omega^2} \tag{54}$$

which thus gives the minimum value of the ratio $\frac{dt}{dt'}$, when the conductivity may be disregarded. It is obvious that $\frac{dt}{dt'}$ must necessarily be positive.

In the general case we find from equations (47), (48) and (51) that

$$\frac{dt}{dt'} = \frac{1}{2}\left\{1 - \frac{2k\nu^2}{(\omega^2+\nu^2)^2} + \frac{1+\frac{2k^2\nu^2}{\omega^2(\omega^2+\nu^2)^2} - \frac{2k(\omega^2+2\nu^2)}{(\omega^2+\nu^2)^2}}{\sqrt{\left(1-\frac{2k}{\omega^2+\nu^2}\right)^2 + \frac{4k^2\nu^2}{\omega^2(\omega^2+\nu^2)^2}}}\right\} \tag{55}$$

where

$$k = N2\pi\frac{e^2}{m}, \tag{56}$$

and

$$\begin{aligned} n^2 &= \tfrac{1}{2}\left\{1-\frac{2k}{\omega^2+\nu^2} + \sqrt{\left(1-\frac{2k}{\omega^2+\nu^2}\right)^2 + \frac{4k^2\nu^2}{\omega^2(\omega^2+\nu^2)^2}}\right\} \\ &= \tfrac{1}{2}\left\{1-\frac{2k}{\omega^2+\nu^2} + \sqrt{1+\frac{4k}{\omega^2+\nu^2}\left(\frac{k}{\omega^2}-1\right)}\right\}. \end{aligned} \tag{57}$$

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Taking into consideration the effect of the earth's magnetic field, by putting

$$\varepsilon_I = 1 - \frac{2k(\omega - h)}{\omega\left(\nu^2 + (\omega - h)^2\right)} \quad \text{and} \quad 2\pi c^2 \frac{\sigma_I}{\omega} = \frac{k\nu}{\omega\left(\nu^2 + (\omega - h)^2\right)} \tag{58}$$

and

$$n_I^2 = \tfrac{1}{2}\left\{1 - \frac{2k(\omega - h)}{\omega\left(\nu^2 + (\omega - h)^2\right)} + \sqrt{1 + 4k\frac{k - \omega(\omega - h)}{\omega^2\left(\nu^2 + (\omega - h)^2\right)}}\right\} \tag{58a}$$

we find

$$\left(\frac{dt}{dt'}\right)_I = \frac{1}{2}\left\{1 - k\frac{2\omega\nu^2 - h\left(\nu^2 + (\omega - h)^2\right)}{\omega\left(\nu^2 + (\omega - h)^2\right)^2} + \frac{1 + k\dfrac{\omega\nu^2(3h - 4\omega) + \omega(\omega - h)^2(3h - 2\omega) + 2k\left(\nu^2 - h(\omega - h)\right)}{\omega^2\left(\nu^2 + (\omega - h)^2\right)^2}}{\sqrt{1 + 4\dfrac{k}{\omega^2}\cdot\dfrac{k - (\omega - h)\omega}{\nu^2 + (\omega - h)^2}}}\right\} \tag{59}$$

The corresponding expression for $\left(\frac{dt}{dt'}\right)_{II}$ is obtained by substituting $-h$ for h in (59).

For $\nu = 0$ we have

$$\left(\frac{dt}{dt'}\right)_{I\,(\nu=0)} = 1 + k\frac{h}{\omega(\omega - h)^2} \quad \text{and} \quad \left(\frac{dt}{dt'}\right)_{II\,(\nu=0)} = 1 - k\frac{h}{\omega(\omega + h)^2}. \tag{60}$$

This formula of approximation for $\left(\frac{dt}{dt'}\right)_I$ cannot be used when $\omega = h$, but in that case (59) is reduced to:

$$\left(\frac{dt}{dt'}\right)_{I\,(\omega=h)} = \frac{1}{2}\left\{1 - \frac{k}{\nu^2} + \frac{\omega^2\nu^2 + 2k^2 - k\omega^2}{\omega^2\nu^2\sqrt{1 + 4\dfrac{k^2}{\omega^2\nu^2}}}\right\}. \tag{61}$$

Correspondingly we find for

$$\varepsilon_\perp = 1 - 2k\frac{\nu^2 + \omega^2 - h^2}{(\nu^2 + \omega^2 + h^2)^2 - 4\omega^2h^2} \quad \text{and} \quad \frac{2\pi c^2}{\omega}\sigma_\perp = \frac{k}{\omega}\cdot\frac{\nu(\nu^2 + \omega^2 + h^2)}{(\nu^2 + \omega^2 + h^2)^2 - 4\omega^2h^2} \tag{62}$$

that

$$n_\perp^2 = \tfrac{1}{2}\left\{1 - 2k\frac{\nu^2 + \omega^2 - h^2}{(\nu^2 + \omega^2 + h^2)^2 - 4\omega^2h^2} + \sqrt{1 + 4k\frac{k(\nu^2 + \omega^2) - \omega^2(\nu^2 + \omega^2 - h^2)}{\omega^2\left[(\nu^2 + \omega^2 + h^2)^2 - 4\omega^2h^2\right]}}\right\}, \tag{63}$$

and that

$$\left(\frac{dt}{dt'}\right)_\perp = \frac{1}{2}\left\{1 - 2k\frac{(\nu^2 - h^2)\left[(\nu^2 + \omega^2 + h^2)^2 - 4\omega^2h^2\right] + 8\nu^2\omega^2h^2}{\left[(\nu^2 + \omega^2 + h^2)^2 - 4\omega^2h^2\right]^2} + \frac{1 + 2k\dfrac{k(\nu^2 + 2\omega^2) - \omega^2(2\nu^2 + 3\omega^2 - 2h^2)}{\omega^2\left[(\nu^2 + \omega^2 + h^2)^2 - 4\omega^2h^2\right]} - 4k\dfrac{(\nu^2 + \omega^2 - h^2)\left[k(\nu^2 + \omega^2) - \omega^2(\nu^2 + \omega^2 - h^2)\right]}{\left[(\nu^2 + \omega^2 + h^2)^2 - 4\omega^2h^2\right]^2}}{\sqrt{1 + 4k\dfrac{k(\nu^2 + \omega^2) - \omega^2(\nu^2 + \omega^2 - h^2)}{\omega^2\left[(\nu^2 + \omega^2 + h^2)^2 - 4\omega^2h^2\right]}}}\right\}. \tag{64}$$

For $\nu = 0$ (63) becomes:

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$$n^2_{\perp(v=0)} = 1 - \frac{2k}{\omega^2 - h^2}, \tag{65}$$

while (64) is reduced to:

$$\left(\frac{dt}{dt'}\right)_{\perp(v=0)} = 1 + \frac{2kh^2}{(\omega^2 - h^2)^2}. \tag{66}$$

For $\omega = h$ (64) is reduced to:

$$\left.\begin{aligned} \left(\frac{dt}{dt'}\right)_{\perp(\omega=h)} = \frac{1}{2}\Bigg\{ & 1 - 2k\frac{v^4+3v^2\omega^2+4\omega^4}{v^2(v^2+4\omega^2)^2} \\ & + \frac{1+2k^2\dfrac{v^4+4v^2\omega^2+6\omega^4}{\omega^2 v^2(v^2+4\omega^2)^2} - 2k\dfrac{2v^4+7v^2\omega^2+4\omega^4}{v^2(v^2+4\omega^2)^2}}{\sqrt{1+4k\dfrac{k(\omega^2+v^2)-v^2\omega^2}{\omega^2 v^2(v^2+4\omega^2)}}}\Bigg\}. \end{aligned}\right\} \tag{67}$$

The above formulas enable us to determine the time required for an impulse to traverse the curved portion of the path where the refractive index is varying. For certain deductions to be made later it may also be of interest to determine the group-velocity for a wave moving in a medium with constant refractive index. This is the case for instance when a ray moves parallel to the surface of the earth for some considerable distance. From the equations (46) and (51) it follows that the group-velocity is determined by

$$u = \frac{c}{n + \omega\dfrac{dn}{d\omega}} = \frac{nc}{n^2 + \frac{1}{2}\omega\dfrac{d(n^2)}{d\omega}} = \frac{nc}{\left(\dfrac{dt}{dt'}\right)}. \tag{68}$$

For $v \rightarrow 0$ we have then the following formulas of approximation:

$$u = nc; \tag{69}$$

$$u_I = \frac{n_I c}{1 + \dfrac{kh}{\omega(\omega - h)^2}} = c\frac{\sqrt{1 - \dfrac{2k}{\omega(\omega - h)}}}{1 + \dfrac{kh}{\omega(\omega - h)^2}}; \quad u_{II} = c\frac{\sqrt{1 - \dfrac{2k}{\omega(\omega + h)}}}{1 - \dfrac{kh}{\omega(\omega + h)^2}}; \tag{70}$$

$$u_\perp = \frac{n_\perp c}{1 + \dfrac{2kh^2}{(\omega^2 - h^2)^2}} = c\frac{\sqrt{1 - \dfrac{2k}{\omega^2 - h^2}}}{1 + \dfrac{2kh^2}{(\omega^2 - h^2)^2}}. \tag{71}$$

4. *The Number of Wave Lengths in the Curved Portion of the Path.*

In the cases mentioned below it is attempted by means of phase-measurements to determine the height of the ionized portion of the atmosphere and it is of importance to know the number of wave lengths in the curved portion of the path. Where the refractive index is a simple function of the height,

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this number may be determined accurately by calculation, but in practice a simple approximation will frequently be preferred. In Fig. 20 *pqr* represents the curved portion of the path of the ray, and the line *ab* indicates the original direction of the ray, which at the surface of the earth, where the refractive index n is equal to unity, forms the angle φ_0 with the vertical.

If the path element ds forms the angle φ with the vertical, then its projection dl on the line *ab* will be determined by

$$dl = ds \cdot \cos(\varphi - \varphi_0), \tag{72}$$

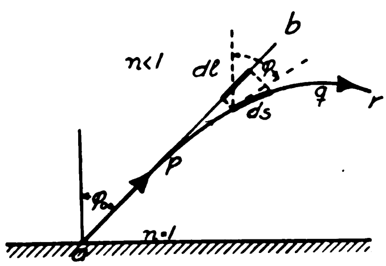

Fig. X. 20. *apqr* is the Path of a Ray starting from the surface of the earth under the angle $\frac{\pi}{2} - \varphi_0$.

where

$$n \sin \varphi = \sin \varphi_0. \tag{73}$$

Between the wave length λ_0 at the earth's surface and λ at the height h, we have the relation

$$\frac{\lambda_0}{\sin \varphi_0} = \frac{\lambda}{\sin \varphi}. \tag{74}$$

The equations (72), (73) and (74) result in:

$$\frac{dl}{\lambda_0} = \frac{ds}{\lambda} \cdot \frac{\sin \varphi \cdot \cos(\varphi - \varphi_0)}{\sin \varphi_0} = \frac{ds}{\lambda} \cdot \frac{\sin^2 \varphi_0 + \cos \varphi_0 \sqrt{n^2 - \sin^2 \varphi_0}}{n^2}. \tag{75}$$

For $n = 1$ we have obviously:

$$\frac{dl}{\lambda_0} = \frac{ds}{\lambda}$$

but this same equation is also valid at the highest point of the path, for which we have $n = \sin \varphi_0$.

Putting then

$$\frac{dl}{\lambda_0} = \beta \frac{ds}{d\lambda} \tag{76}$$

and, consequently,

$$\beta = \frac{\sin^2 \varphi_0 + \cos \varphi_0 \sqrt{n^2 - \sin^2 \varphi_0}}{n^2} \tag{77}$$

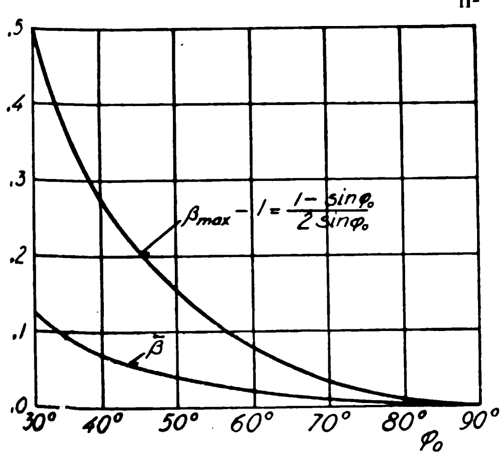

Fig. X. 21. The value of $\beta_{max} - 1$ and of $\bar{\beta}$ as a function of φ_0 for $30^0 \leqq \varphi_0 \leqq 90^0$.

we find easily that β assumes its maximum value, viz:

$$\beta_{max} = \frac{\cos^2 \varphi_0}{2 \sin \varphi_0 (1 - \sin \varphi_0)} \tag{78}$$

for

$$n^2 = 2\,\mathrm{tg}^2 \varphi_0 (1 - \sin \varphi_0).$$

Fig. 21 shows the curve

$$(\beta_{max} - 1) = \frac{1 - \sin \varphi_0}{2 \sin \varphi_0}$$

for

$$30^0 \leqq \varphi_0 \leqq 90^0.$$

Since β is equal to unity both at the lowest and at the highest point of the ray-path the mean value of β will be much nearer to unity than to β_{max}. We estimate, instead of (76), that:

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$$\frac{dl}{\lambda_0} = (1 + \bar{\beta}) \frac{ds}{\lambda}, \tag{79}$$

where

$$\bar{\beta} = \tfrac{1}{4} (\beta_{max} - 1) = \frac{1 - \sin \varphi_0}{8 \sin \varphi_0}. \tag{80}$$

This value of $\bar{\beta}$ is shown in Fig. 21.

From this it appears that for $\varphi_0 > 30^0$ the number of wave lengths in the half ray path *Apq*, Fig. 22, will be only inappreciably smaller than $\frac{Aq'}{\lambda_0}$, which in its turn is slightly smaller than $\frac{Aq}{\lambda_0}$.

In assuming therefore, that the number N_a of wave lengths in the ray reverting from the upper atmosphere is

$$N_a = \frac{Aq + qB}{\lambda_0} = \frac{2Aq}{\lambda_0}, \tag{81}$$

we thus use a value which is slightly too high, but the error is generally not very great, and the approximation (81) is much better than the one obtained by putting

$$N_a = \frac{Ac + cB}{\lambda_0} = \frac{2Ac}{\lambda_0},$$

this value being much too high.

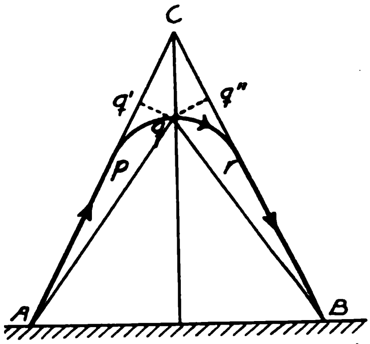

Fig. X. 22. The Actual Ray Path is *ApqrB*. The Number of Wave Lengths in this is approximately equal to $\frac{2Aq}{\lambda_0}$, λ_0 being the wave length corresponding to $n = 1$.

5. *Attenuation of a Ray Travelling along a Curved Path.*

For a ray leaving the surface of the earth under an angle ψ, we have already shown how to determine the maximum height h_0 to which the ray will reach, and the radius of curvature ϱ at this point. As shown, the path may then with good approximation, in some cases, be replaced partly by the parabola *CED* in Fig. 23, with the equation $x^2 = 2\varrho y$, having the radius of curvature ϱ at the point of origin, and partly by the two tangents *AC* and *DB* forming the angle ψ with the surface of the earth.

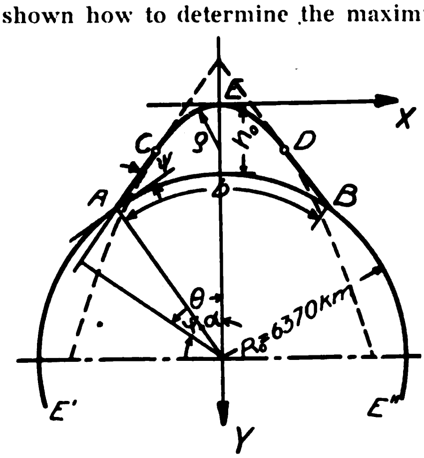

Fig. X. 23. Parabola Approximation for the Path of the Ray.

From the γ_0-curves we now determine the value of γ_0 at the height h_0, and approximate the γ_0-curve by a straight line, which in the logarithmic system of coordinates used for γ_0 corresponds to putting

$$\gamma = \gamma_0 \, e^{-\beta y} = \gamma_0 \, 10^{-\beta_{10} y} \tag{82}$$

where β_{10} may be easily determined from the graphical representation, and where $\beta = 2.303 \cdot \beta_{10}$.

The amplitude of the ray traversing the curved portion of the path will be attenuated at the rate of $e^{-\Gamma}$, where

$$\Gamma = \int_{ACEDB} \gamma \, ds = \gamma_0 \int_{ACEDB} e^{-\beta y} \, ds = 2\gamma_0 \int_{EDB} e^{-\beta y} \, ds = \gamma_0 \, \varrho K, \tag{83}$$

K being a function of $a = \frac{\beta \varrho}{2}$ and of $\theta = \alpha + \psi$ (as mentioned above, α may be determined by means of the abac in Fig. 13).

If the path is of parabolic shape over the entire portion where attenuation occurs, then we have:

$$K = \frac{2}{\varrho} \int e^{-\beta y} \, ds = \frac{2}{\varrho} \int_0^{\infty} e^{-\beta y} \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx = 2 \int_0^{\infty} e^{-a z^2} \sqrt{1 + z^2} \, dz \tag{84}$$

where $z = \frac{x}{\varrho}$ and $a = \frac{\beta \varrho}{2}$.

In (84) we may replace $\sqrt{1 + z^2}$ by $z + e^{-0.9 z}$, without thereby making an error of more than about 2%, and the integration in (84) may then be performed without difficulty.

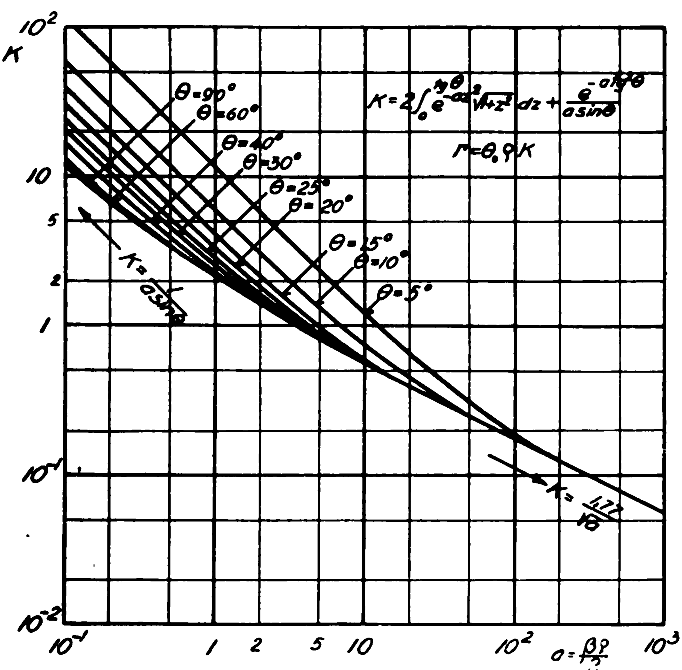

Fig. X 24. The Values of K as Function of $a = \frac{\beta \varrho}{2}$ and $\theta = \alpha + \psi$.

If ψ is small, a loss may occur also along the rectilinear pieces *AC* and *DB*, and, the co-ordinates of *D* being $(x, y) = \left(\varrho \operatorname{tg} \theta, \frac{\varrho}{2} \operatorname{tg}^2 \theta\right)$, we find in this case that *K* is determined by:

$$K = 2\int_0^{\operatorname{tg} \theta} e^{-az^2} \sqrt{1+z^2}\, dz + \frac{2}{\varrho} \int_{\frac{\varrho}{2} \operatorname{tg}^2\theta}^{\infty} e^{-\beta y} \frac{dy}{\sin\theta} = 2\int_0^{\operatorname{tg} \theta} e^{-az^2} \sqrt{1+z^2}\, dz + \frac{e^{-a \operatorname{tg}^2 \theta}}{a \sin\theta} \cdot \quad (85)$$

Using the above mentioned approximation for $\sqrt{1+z^2}$ we find the following formula of approximation for *K*:

$$K \cong \frac{1}{a}\left\{1 + \frac{1-\sin\theta}{\sin\theta} e^{-a \operatorname{tg}^2\theta} + \sqrt{\pi a}\, e^{\frac{0.2025}{a}} \left(I_{\left(\operatorname{tg}\theta \sqrt{a} + \frac{0.45}{\sqrt{a}}\right)} - I_{\left(\frac{0.45}{\sqrt{a}}\right)}\right)\right\} \quad (86)$$

where

$$I_{(t)} = \frac{2}{\sqrt{\pi}} \int_0^t e^{-x^2} dx.$$

Fig. 24 shows the values of *K* for various values of *a* and of θ.

It must be noticed, however, that the attenuations calculated in this way are too small because, according to *Gans*[1], stationary oscillations will be formed above the plane where $\cos\psi_0 = n$ and where total reflection should occur according to the geometrical optics. These stationary oscillations may cause considerable losses which in general will increase with decreasing frequency and for long waves be very great. But as the theory of *Gans* deals only with the simplified case where the conductivity is equal to zero, we cannot use his formulas and shall not enter further into this problem. For very long waves we assume Γ to have twice the value determined by formulas (83) to (86).

[1] *R. Gans:* Ann. d. Physik IV Bd. 47, p. 709—736. 1915.

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CHAPTER XI.

THE PROPAGATION OF RADIO WAVES. COMPARISON BETWEEN THEORY AND EXPERIENCE.

1. *Introduction.*

The question which we shall first try to answer is this: Can a theory of radio transmission be based solely on the solar ionization and on the recombination between the electrons and the positive ions?

In order to clarify this problem we shall quote a few of the most reliable results gained from the experiences of the transmission of radio waves and bearing on this subject.

It has been proved that radio waves of wave lengths greater than a certain limiting value, $\lambda_{RD\,min}$, in the daytime return regularly from the upper atmosphere to the surface of the earth at some distance from the transmitting station. $\lambda_{RD\,min}$ is called the minimum length of returning daytime waves. We shall also call it the regular short-wave limit for day transmission. Correspondingly there is a minimum length of returning nighttime waves, $\lambda_{RN\,min}$, which we shall also call the regular short-wave limit for night transmission. By daytime we understand in this connection midday in the summer at a northern latitude of about 40 degrees. By nighttime we understand a winter night 6 to 10 hours after sunset.

The values of $\lambda_{RD\,min}$ and $\lambda_{RN\,min}$ have not yet been determined with any great precision. We will take

$$\lambda_{RD\,min} = 8.5\text{ m} \quad \text{and} \quad \lambda_{RN\,min} = 18.9\text{ m}.$$

These values will hardly be the final ones; but on the other hand the values chosen will not — at least in our opinion — differ from the true values enough to cause any serious error in the following considerations.

In order to justify — or at least explain — these values of the wave length limits we may give the following evidence.

The *Telefunken* transmissions between Nauen and Buenos Aires[1] show that the 18 m wave sometimes, but not always, is effective by night while the 26 m wave normally seems always to be effective at night. Between Nauen and Iwatsuki, Japan, the 16 m wave is not always effective by night, while this is the case with the 25.1 m wave[2].

Between New York and Buenos Aires the 15 m wave according to *E. F. W. Alexanderson* is regularly effective by day but not by night[3]. Considering this

[1] *H. Rukop:* Jahrb. d. drahtl. Tel. Bd. 28, p. 41—50. 1926.

[2] *Toyokichi Nakagami:* Proc. Imp. Acad. Tokyo. Vol. 2, No. 7, p. 338—340. 1926.

[3] »Q. S. T.« April 1927, p. 15—16.

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and other evidence we are of the opinion, that the value of $\lambda_{RN\,min}$ must be in the neighbourhood of 18—19 m [1].

With regard to the value of $\lambda_{RD\,min}$ the available evidence is less reliable. We are not aware of any instance in which waves for which $\lambda < 8$ m have been found to return regularly from the upper atmosphere, even if some cases are known in which such waves have been received over considerable distances[2 3]. As mentioned above the 15 m wave is regularly effective by day between New York and Buenos Aires; and even at the time of sunset and sunrise this wave seems to be effective in most cases, but not so by night. *Telefunken* has succeeded, by means of a 15 m wave, in transmitting around the earth and back to the transmitting station, and this wave, therefore, must be effective during the greatest part of the 24 hours of the day[4]. $\lambda_{RD\,min}$ therefore must be considerably less than 15 m. Furthermore, according to *T. L. Eckersley*[5], a 10 m wave may give good transmission by day between England and Sydney. We therefore conclude that $\lambda_{RD\,min}$ must lie somewhere between 8 and 10 m and we have provisionally assumed $\lambda_{RD\,min} = 8.5$ m.

The other experimental fact, which we shall make use of, is that even for waves as long as 5000 m the conditions of transmission change very quickly at the time of sunset or sunrise. This fact is well known and it will suffice to refer to the work of *Lloyd Espenschied, C. N. Anderson* and *Austin Bailey* on Transatlantic radio telephone transmission[6].

We shall next consider the conclusions to be drawn from these facts and here, provisionally, assume that the minimum value of the refractive index n both for the 8.5 m wave by day and the 18.9 m wave by night is to be found at altitudes between 90 and 160 km. We shall later prove that the height of maximum ionization by day cannot be less than about 90 km. We furthermore assume that a tolerably regular transmission demands that returning rays must leave the surface of the earth at angles not less than 8^0 to 10^0 but this assumption is not of great importance. According to Fig. X. 14 the reduction Δn_0 in the refractive index must then be $\Delta n_0 \simeq 0.03$, and the minimum value of the refractive index therefore not greater than $1 - 0.03 = 0.97$.

If $\Delta n < \Delta n_0 = 0.03$ then the $\lambda_{RD\,min}$-wave will not return to the surface of the earth. Even if $\Delta n > \Delta n_0$, and consequently $n < 1 - \Delta n_0 = 0.97$, there is

[1] Quite recently *T. L. Eckersley:* (»Experim. Wireless«, p. 213—222, April 1927: Abstract of Paper read before the Wireless Section, I. E. E., on March 2nd, 1927) states that the short-wave limit for night-transmission probably is somewhere in the neighbourhood of 17 m. This is in rather good agreement with $\lambda_{RN\,min}$ being equal to 18.9 m.

[2] According to »Q. S. T«. March 1927, p. 35, communication was established at a wave length of 5.19 m between two stations about 50 km apart and across New York City. This is probably due at least partly to refraction and scattering at the metal parts of the sky-scrapers of New York. See also section 2 of this chapter.

[3] *T. L. Eckersley:* l. c.

[4] Paper read before the Berlin Academy on December 16th 1926 by *K. W. Wagner.* (»Forschungen und Fortschritte«, January 20th, 1927). *E. Quäck*: Jahrb. d. drahtl. Tel. Bd. 28, p. 177—178. 1926.

[5] *T. L. Eckersley*: l. c., Fig. 38.

[6] *L. Espenschied, C. N. Anderson* and *A. Bailey*: Proc. Inst. Rad. Eng. Vol. 14, p. 7—56. 1926.

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still the possibility that the waves for which $\lambda \geqq \lambda_{RD\,min}$ return to the earth, while the waves for which $\lambda < \lambda_{RD\,min}$ will penetrate so deeply into the ionized layer that they are too strongly attenuated to reach the earth with any appreciable intensity. We shall have to return to this question later. For the present it suffices to know that Δn_0 cannot be less than 0.03.

With such a small value of Δn and for such high frequencies as $\omega \geqq 10^8$ the conductivity σ will have very little influence on the value of Δn which therefore depends practically only upon the value of the dielectric constant $\varepsilon = 1 - \Delta\varepsilon$, *i.e.* upon the value of $\Delta\varepsilon$ (See Fig. VIII. 6).

We therefore have

$$n = 1 - \Delta n_0 \cong \sqrt{\varepsilon} = \sqrt{1 - \Delta\varepsilon_0} \text{ or } \Delta\varepsilon_0 \simeq 2\Delta n_0 = 0.06. \tag{1}$$

We need not in this consideration take any account of the influence of the magnetic field of the earth on the values of σ and $\Delta\varepsilon$, since at such high frequencies this influence is altogether negligible; see Figs. VII. 7 and 8. Furthermore we need only consider the influence of the electrons on σ and $\Delta\varepsilon$, since we provisionally assume that the electrons are lost solely by recombination with positive ions. The density of electrons and ions must then necessarily be the same and the influence of the ions upon σ and $\Delta\varepsilon$ will therefore, according to Chapter VI, be only about a 50000$^{\text{th}}$ part of that of the electrons.

We may therefore (see Chapter VI, Fig. 3 and formula (38)) put

$$\Delta\varepsilon \simeq 4\pi \frac{e^2}{m} \cdot \frac{1}{\omega^2} \sim 3.2 \cdot 10^9 \cdot \frac{1}{\omega^2} \cdot \tag{2}$$

The smallest possible value n_{eN} of the greatest electron density by night is consequently determined by

$$n_{eN} = \frac{0.06}{3.2 \cdot 10^9 \dfrac{1}{10^{16}}} = 1.9 \cdot 10^5 \text{ electrons per c c,} \tag{3}$$

and the maximum electron density by day must at least be equal to

$$n_{eD} = \frac{18.9^2}{8.5^2} \cdot n_{eN} = 4.9\, n_{eN} = 9.2 \cdot 10^5 \text{ electrons per c c.} \tag{4}$$

The altitudes of maximum electron density by day and night are denoted by h_D and h_N respectively, and the electron density by day at the altitude h_N is called n'_{eo}. Accordingly we have the following relations:

$$\frac{n'_{eo}}{n_{eN}} = 4.9 \text{ for } h_D = h_N \text{ and } \frac{n'_{eo}}{n_{eN}} < 4.9 \text{ for } h_D \gtrless h_N. \tag{5}$$

For the present, taking into consideration recombination alone we have according to Chapter V formula (6):

$$n_{eN} = \frac{n'_{eo}}{1 + \alpha n'_{eo} \cdot t} = \frac{n_{eD}}{4.9}, \tag{6}$$

where $t = 10$ hours $= 3.6 \cdot 10^4$ seconds and α is the recombination constant.

From (6) we get:

$$\alpha t = \frac{4.9}{n_{eD}} - \frac{1}{n'_{eo}} \leqq \frac{3.9}{n_{eD}}, \tag{7}$$

$$\alpha t \leqq 4.25 \cdot 10^{-6} \quad \text{or} \quad \alpha \leqq 1.2 \cdot 10^{-10}. \tag{8}$$

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For an air pressure of $p = 760$ mm the value of α is about $1.6 \cdot 10^{-6}$, see Chapter V (8).

Espenschied, Anderson and *Bailey* in their above mentioned investigations used a 5270 m wave corresponding to $\omega \simeq 3.6 \cdot 10^5$. For such a wave one electron per c c at an altitude of 100 km will very nearly give a value of $\Delta\varepsilon$ equal to 0.03 and at an altitude of 80 km $\Delta\varepsilon \simeq 0.001$, see Chapter VI. Fig. 3.

It will, therefore, not be possible for this wave to reach altitudes at which the electron density is greater than 10^8 electrons per c c. Considering the above mentioned values of the recombination constant, *i. e.* $\alpha \leqq 1.2 \cdot 10^{-10}$ at an altitude of h_N and $\alpha = 1.6 \cdot 10^{-6}$ at the surface of the earth, and also considering the evidence given in Chapter V (see f. inst. Fig. V. 3) we may safely conclude that the recombination constant is not greater than 10^{-8} at the highest altitudes to which the $\omega = 3.6 \cdot 10^5$ wave may reach.

The electron density in that part of the atmosphere which is ›open‹ to the wave cannot alter more rapidly than according to the following formula:

$$n_{et} = \frac{\text{const.}}{1 + 10^{-8} \cdot 10^3 \cdot 3.6 \cdot 10^3 T} = \frac{\text{const.}}{1 + 0.036 T}, \tag{9}$$

where T is the time, measured in hours, after ionization has ceased, *i. e.* after sunset. (See Chap. V, formula (6)).

The electron density — and therefore also the ion density — will according to (9) vary so slowly that it will be impossible to explain on this basis the rather sudden variations in the propagation of the waves about sunset or sunrise.

We therefore conclude that it will not be possible to explain the propagation of radio waves in a satisfactory manner if we assume that free electrons can be lost only by recombination with positive ions.

Relatively few of the numerous authors who have discussed the theory of the propagation of radio waves have paid any great attention to the possible state of ionization in the upper atmosphere and none of them have treated this important problem thoroughly. Most authors simply assume the existence of that state of ionization which seemed most suitable to explain the special problem investigated. This is the case in the very important papers of *H. J. Round, T. L. Eckersley, K. Tremellen* and *F. C. Lunnon*[1], *A. Hoyt Taylor* and *E. O. Hulburt*[2], *E. O. Hulburt*[3], *W. G. Baker* and *C. W. Rice*[4], *A. Hoyt Taylor*[5], *G. Breit* and *M. A. Tuve*[6], *R. Mesny*[7] and *T. L. Eckersley*[8]. Since these papers hardly enter into a discussion of the ionization problem they cannot contribute substantially to a solution of the question under discussion.

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1 *H. J. Round, T. L. Eckersley, K. Tremellen* and *F. C. Lunnon*: Journ. Inst. E. E. Vol. 63, p. 933—1001, 1925.

2 *A. Hoyt Taylor* and *E. O. Hulburt*: Phys. Rev. (II) Vol. 27, p. 189—215, 1926.

3 *E. O. Hulburt*: Journ. Frankl. Inst. Vol. 201, p. 597—634, 1926.

4 *W. G. Baker* and *C. W. Rice*: Refraction of Short Radio Waves in the Upper Atmosphere. Am. Inst. E. E. (Abridgment in: Journ. Am. Inst. E. E., Vol. 45, 535—539, 1926).

5 *A. Hoyt Taylor*: Proc. Inst. Rad. Eng. Vol. 14, p. 521—540, 1926.

6 *G. Breit* and *M. A. Tuve*: Phys. Rev. (II) Vol. 28, p. 554—575, 1926.

7 *R. Mesny*: Les ondes électriques courtes. Paris. 1927.

8 *T. L. Eckersley*: Experim. Wireless. Vol. 4, p. 213—222, 1927. (Abstract).

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There are, however, some authors, to be mentioned below, who have entered somewhat further into a discussion of the ionization in the upper atmosphere and who have therefore met with the difficulty stated above. They have tried to escape this difficulty in various ways of which we may mention the following:

G. J. Elias[1] points out ((a) p. 357) that recombination cannot account for the rapid variations about sunset and mentions some possibilities of escaping this difficulty, but does not seem to be quite satisfied with any of them. He also is of the opinion that the ionization caused by the ultra-violet radiation from the sun in the daytime will decrease so rapidly by night, that this ionization cannot explain the wave propagation by night ((b) p. 66; see also Chapter V, note 2 p. 69). In order to overcome this difficulty this author assumes two layers of ionization: An upper layer permanently ionized by a corpuscular radiation (α-particles) from the sun and a lower layer caused by the ultra-violet radiation from the sun and effective only in the day time. We cannot accept this point of view. The corpuscular radiation from the sun, which causes the aurora polaris, is in our opinion far too erratic to be the chief determining factor in the propagation of short waves by day and night and of longer waves by night. (See also Chapter V. sects. 6 and 8). As mentioned in Chapter V there can — at least in our opinion — be no doubt whatever as to the ultra-violet radiation from the sun being the main source of that ionization in the upper atmosphere which determines the normal and regular propagation of the radio waves. (Furthermore, as mentioned in Chapter IV, we cannot agree with this author's point of view concerning the composition of the upper atmosphere).

T. L. Eckersley[2] tries to prove that recombination alone will give a much more sharply defined lower edge of the ionization-layer by night than by day and thereby explains some of the differences between day and night transmission. But there must probably be some mistake in this, because under the assumptions made by this author the steepest gradient in the density of electrons (and ions) will always be steeper by day than by night[3].

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[1] *G. J. Elias*: (a). E. N. T. Bd. 2, p. 351—358, 1925.
(b). Jahrb. d. drahtl. Tel. Bd. 27, p. 66—73, 1926.

[2] *T. L. Eckersley*: Rad. Rev. Vol. 2, 64—65, 1921.

[3] *G. J. Elias* has called the attention to this error (l. c. (a) p. 357) which may be proved thus: *Eckersley* assumes the recombination constant to be proportional to the pressure, $\alpha = \alpha_0 \cdot e^{-\frac{h}{H}}$; the stationary day value of electron (and ion) density is then, according to Chapter V (4) and (53), determined by

$$n_0 = \sqrt{\frac{K'A}{\alpha_0}} \cdot e^{-\frac{AH}{2} e^{-\frac{h}{H}}} = K \cdot e^{-\frac{AH}{2} e^{-\frac{h}{H}}}.$$

We therefore have

$$\frac{dn_0}{dh} = \tfrac{1}{2} AKe^{-\left(\frac{h}{H} + \frac{AH}{2} e^{-\frac{h}{H}}\right)}$$

and the value of $\frac{dn_0}{dh}$ is maximum for $e^{\frac{h}{H}} = \frac{1}{2} AH$, and

$$\left(\frac{dn_0}{dh}\right)_{max} = \frac{K}{eH}.$$

The value of the electron density t seconds after sunset is

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H. Lassen[1] has probably entered into the discussion of the ionization in the upper atmosphere with regard to its influence on the propagation of radio waves further than any one of the other authors. But he mainly deals with short waves and assumes that the electrons set free by means of the ultraviolet radiation from the sun form ions by adhering to the hydrogen molecules at some of the collisions with the latter which — in his opinion — are present in great number even at high altitudes. He totally neglects the ability of oxygen and water molecules to capture free electrons. His reasons for making these assumptions seem to be: (1) that the ionization caused by the sun would have to be unreasonably small if the electrons remained free, (2) that the numbers of oxygen and water molecules present in the upper atmosphere are so relatively small that these molecules cannot catch any appreciable number of free electrons[2].

The first mentioned reason depends upon the very small value of the recombination constant used by this author, namely $\alpha = 2.2 \cdot 10^{-13}$, and we consider this value too small; with our value, namely $\alpha = 4 \cdot 10^{-11}$, the available ultra-violet radiation is just of the right order of magnitude and by no means too great. (See Chapter V, section 9).

With regard to the second reason mentioned above, it suffices to say that our investigations show the number of oxygen molecules in the upper atmosphere to be amply sufficient to catch the necessary number of free electrons. Possibly it would also be found difficult to explain the rapid variations about sunset on the basis of the assumptions made by *H. Lassen.*

Our main reason, however, for discarding these assumptions in the more elaborate theory given here, is that, as stated in Chapter V, we are forced to consider it very unlikely that the free electrons should be captured by the free, neutral hydrogen molecules in the manner assumed by this author.

In Chapter V we have from physical considerations assumed that electrons moving with small velocities tend to form ions by adhering to oxygen and water molecules at some of the collisions with these molecules, and we have determined the influence of this loss of free electrons on the electron densities by day and by night. We have furthermore shown that this loss of free electrons may alter the electron density very rapidly about sunset, see for example Fig. V. 5. This explains in a satisfactory manner the rapid variations in the wave propagation about sunset. It will also be shown in the following that

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$$n_t = \frac{n_0}{1 + \alpha n_0 t} = \frac{K \cdot e^{-\frac{AH}{2} \cdot e^{-\frac{h}{H}}}}{1 + \alpha_0 K t \cdot e^{-\left(\frac{h}{H} + \frac{AH}{2} \cdot e^{-\frac{h}{H}}\right)}}.$$

Therefore

$$\frac{dn_t}{dh} = \frac{dn_0}{dh} \cdot \frac{1 + \frac{2\alpha_0 K t}{AH} \cdot e^{-\frac{1}{2} AH \cdot e^{-\frac{h}{H}}}}{\left(1 + \alpha_0 K t \cdot e^{-\left(\frac{h}{H} + \frac{AH}{2} \cdot e^{-\frac{h}{H}}\right)}\right)^2} < \left(\frac{dn_0}{dh}\right)_{max}$$

and

$$\left(\frac{dn_t}{dh}\right)_{max} < \left(\frac{dn_0}{dh}\right)_{max}.$$

[1] *H. Lassen:* Jahrb. d. drahtl. Tel. Bd. 28, p. 109—113 139—147. 1926.
[2] l. c., p. 141.

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the further consequences of this assumption are in accordance with the experimental facts.

Therefore we feel justified in stating that a satisfactory theory of the normal and regular propagation of radio waves necessarily must be based upon the following assumptions:

(α) The ionization in the upper atmosphere which is the chief determining factor with regard to the normal and regular propagation of radio waves is mainly due to the ultra-violet radiation of the sun.

(β) The electrons and ions are subject to recombination, the value of the recombination-constant lying within the theoretical limits found in Chapter V.

(γ) The free electrons may be captured also by neutral oxygen and water molecules, but not by nitrogen, helium or hydrogen molecules.

We shall next try to answer this question: Does the experience with regard to the propagation of radio waves provide any means of fixing upper and lower limits of the air pressure existing at altitudes of 110 to 150 km?

As mentioned above the 15 m wave is regularly effective by day between New York and Buenos Aires (about 9000 km) and according to Telefunken's[1] experience the 18 m wave is generally effective by day between Nauen and Buenos Aires (about 12000 km), while the 15 m and 16.2 m waves may travel around the entire earth and back to the transmitting station[2].

The propagation may take place in two different ways:

(1) The paths of the waves, the »rays«, may be refracted in the lower part of the ionized region of the air, where the refractive index varies rapidly with the altitude thereby causing a rather sharp bend at the top of the path. The rays are thus coming back to the earth within a distance of less than — and generally much less than — 2500 km.

(2) The rays may go somewhat higher up in the atmosphere to such altitudes where the refractive index varies relatively slowly and where the radius of curvature at the top of the path therefore is very great, only slightly smaller than the radius of the earth plus the altitude of the top of the path. The rays in this case come back to the earth at distances between 5000 and 40000 km.

In case (1) the ray, in order to reach distances of 10000 to 40000 km, must return to the surface of the earth at least respectively 4 to 16 times and must be reflected respectively 3 to 15 times at the surface of the earth.

For the short waves treated here this reflection will generally cause a very serious loss. Even by reflection at an ideal plane horizontal surface there is a very noticeable loss at great angles of incidence, which are the only angles effective here, at least for such waves for which the electric field is parallel to the plane of incidence, see Chapter VIII. Figs. 11—18. But apart from this there are other even more serious losses to be considered.

Thus, if the surface at the point of reflection is not horizontal, but tilted β degrees against the horizon in the direction of the ray then the wave will not be reflected at the correct angle ψ but at an angle $\psi \pm 2\beta$. Even if the value of β is only a few degrees the reflected ray will in most cases be almost completely lost for the long distance transmission under consideration.

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[1] *H. Rukop:* l. c., Figs. 19 and 20.

[2] *E. Quäck:* Jahrb. d. drahtl Tel. Bd. 28, p. 177—178. 1926.

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If the earth at — or in the vicinity of — the point of reflection is either covered with trees, scrubs, buildings, or the like, or if that part of the earth's surface is hilly or mountainous then a considerable part of the energy in the incident ray will be absorbed, and what is left will be scattered to such a degree that very little of it is reflected in the desired direction. Even the seas and oceans will generally cause great losses at the reflection of these short waves; only when the surface is smooth and without swells will this loss be relatively small.

Considering this we believe it safe to assume that no long distance transmission can be fairly regular or fairly effective if it depends upon more than a few reflections.

We therefore come to the conclusion that regular transmission over very great distances must necessarily be effected according to case (2) and this conclusion is strengthened by the evidence brought forward in the next section, where this question is treated more thoroughly.

In this case the ray for very great transmission distances must travel a long way, say 5 000 to 10 000 km or more, at an altitude, where the value of the refractive index is but little higher than the minimum value, and where accordingly the electron density is only slightly less than the maximum density.

The smallest possible value of the greatest electron density by day is, according to (4), equal to $9.2 \cdot 10^5$ electrons per c c. We may therefore, without committing any great error, take the electron density of the uppermost part of the ray to be equal to $7.5 \cdot 10^5$ electrons per c c. If we assume that the ray travels 8 000 km at this electron density and that the attenuation constant $8000 \cdot \gamma_0$ must not exceed say 8 — *i. e.* the amplitude is attenuated in the ratio of e^{-8} — then the value of γ_0 must not exceed 0.001 corresponding to $\sigma \simeq 5 \cdot 10^{-20}$. Each electron per c c must therefore give a conductivity of not more than $6.7 \cdot 10^{-26}$ e. m. u.

Taking $\omega = 1.25 \cdot 10^8$ ($\lambda = 15.1$ m) the corresponding value of ν, according to formula (22) in Chapter (VI), is determined by

$$2.8 \cdot 10^{-13} \frac{\nu}{\nu^2 + \omega^2} = 2.8 \cdot 10^{-13} \cdot \frac{\nu}{\nu^2 + 1.6 \cdot 10^{16}} = 6.7 \cdot 10^{-26},$$

or

$$\nu \leqq 3.8 \cdot 10^3. \tag{10}$$

The value of ν corresponding to the atmosphere F at an altitude of about 125 km is about $3.2 \cdot 10^4$ *i. e.* almost 10 times greater (see Chapter IX. Fig. 3).

We may therefore conclude that the actual pressure at an altitude of about 125 to 150 km at all events cannot be higher than that of atmosphere F, and we thus come to the further conclusion:

(δ) The number of collisions per sec. suffered by an electron at an altitude of 125 to 150 km cannot be greater than about $3.8 \cdot 10^3$ and the air pressure at these altitudes cannot be higher than the pressure in the atmosphere F.

This result indicates that the air pressure at these altitudes ought to be considerably less than in the atmospere F. But in order to get air pressures appreciably below those of atmosphere F, we must either assume a temperature of the stratosphere considerably below -54^0 C or else we must reduce the amount of helium below the value here assumed, namely 0.0004 per cent by volume. At present there is no other evidence for either of these assumptions

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but since the evidence gained by the study of meteors makes it very unlikely that the mass density should be considerably less than in the atmosphere *F*, we think it would make the present theory weaker if it was found necessary to assume the actual air pressures to be considerably lower than those of atmosphere *F*. But happily this is not so, and we are able to base the following considerations upon the atmosphere *F*.

We have been able to do so because up to the present we have — for the sake of simplicity — assumed the mean free path l_{el} of the electrons in the atmosphere *F* to be the same as it would have been if this atmosphere consisted solely of nitrogen of the same total pressure as *F*. In fact the atmosphere *F* at these altitudes consists almost entirely of helium and the free path for slowly moving electrons is considerably greater in this gas than in nitrogen. We shall here assume $l_{el,He}$ to be about 10 times greater than $l_{el,N}$ (see p. 42 and 144 and Fig. IX 1 in the appendix) and with this value of l_{el} the value of v in the atmosphere *F* at an altitude of about 125 km will be about $3.2 \cdot 10^3$, *i. e.* a little less than the maximum value given in (10). We have previously used the symbol *F'* for this atmosphere.

Having thus justified our choice of the atmosphere *F'* and the maximum value of v_{el} at altitudes between 125 and 150 km, we may then go one step further.

Experience shows that waves for which $\lambda > 24$ m generally do not give regular long distance transmission in the daytime, compare f. inst. *H. Rukop's* paper[1]. But experience also shows that wave lengths a little below 19 m, f. inst. 18 m[2] may come through during daytime. We must therefore conclude that for a wave length somewhere between 18 and 24 m the attenuation by day at the top of the path is just too great to allow of effective long distance transmission.

With the same length of the top-path as assumed above, namely 8 000 km, and with the same value of v_{el} as found above, namely $3.8 \cdot 10^3$, we get the total attenuation constant for the 24 m wave to be equal to[3]

$$8 \cdot \left(\frac{24}{15.1}\right)^2 \simeq 20, \tag{11}$$

i. e. the amplitudes would in this case be reduced in the ratio e^{-20}.

For a 19 m wave the corresponding attenuation would be

$$8 \cdot \left(\frac{19}{15.1}\right)^2 \simeq 12{,}6. \tag{12}$$

These attenuations are just of the right order of magnitude. If we supposed the value of v_{el} to be only half the maximum value given in (10), we should get only half of the attenuation constant found in (11) and (12) and these values, 10 and 6.3, would presumably be a little too small[4].

We may therefore conclude that:

(ε) At altitudes between 125 and 150 km the value of v_{el} must be about $2 \cdot 10^3$ to $4 \cdot 10^3$. On account of the comparatively long free paths for slowly moving electrons in helium v_{el} for the atmosphere *F'* must be expected to have a value within these limits.

---

[1] *H. Rukop*: l. c., Figs. 17 to 21. [2] l. c., Figs. 19 and 20.

[3] Because $\gamma \simeq \frac{2\pi\sigma}{\sqrt{\varepsilon}} \simeq k_1\sigma$ (Chapt. VIII, (8)), where $\sigma \simeq \frac{k_2}{\omega^2}$ (Chapt. VI, (25)).

[4] It should be remembered that (11) and (12) give the absolute maxima of the attenuation, namely the summer noon values.

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Some of the questions dealt with in the above will be treated more thoroughly in the following. We have made this somewhat cursory review mainly in order to justify our fundamental assumptions[1].

2. *Propagation of Short Waves.*

(a). General Considerations and Remarks.

We do not intend to discuss at any length the influence on the transmission of the shape, orientation, or height of the transmitting or receiving aerials. These important questions no doubt deserve to be treated very thoroughly, but such a discussion would fall outside the scope of the present work. It is necessary, however, to make just a few remarks concerning these questions.

We shall generally assume the transmitting aerial to be vertical and to be radiating in all horizontal directions. We shall further assume, that the radiation in a direction forming an angle ψ with the surface of the earth does not alter very much with the value of ψ, at least as long as ψ is below 30^0 degrees.

We shall consider neither transmission from horizontal aerials nor highly directive transmitting systems (»Beam«-systems). We must point out, however, that experience has shown that even the beam systems with reflectors do not concentrate the whole radiation energy within the »Beam«, but radiate freely in most other directions too. Thus the English *Marconi* beam station sending to Canada is received with good intensity over very large areas east of England.

The *Telefunken* experiments with a horizontal aerial and a parabolic reflector also show strong radiation in the backward direction[2]. We do not see anything surprising in this, however, since a radiating system having a reflector of which the greatest dimensions are only about one wave length necessarily must radiate in many — not to say all — directions, but we find it necessary to point this out in order to avoid unjustified conclusions based on the assumption that such radiating systems radiate only in one single direction.

Bearing this in mind and further considering the fact that a horizontal aerial — at least within large horizontal angles — will also radiate in the manner assumed here, we are brought to the conclusion that the limitation set forth in the beginning of this section with regard to the type of transmitting aerial is not a very serious one. Most of the results found in the following will, no doubt, also apply to other types of aerials, at least with some slight and natural alterations[3].

For a vertical aerial of height $\frac{1}{4}\lambda$ the theory of *Sommerfeld*, see Chapter III, sect. 3, may be applied to the transmission problem in the neighbourhood of the transmitter. It appears from Fig. III. 6 that the critical distance — corresponding to the numerical distance $\varrho = 1$ in *Sommerfeld*'s theory — for the short waves considered at present will be about 0.1 km. For distances greater than

[1] A full list of publications bearing on the problem of radio wave propagation is to be found in: *A. Sacklowski:* E. N. T. Bd. 4, p. 31—74. 1927.

[2] *A. Meissner:* Telefunken Zeitung. VIII. Jahrg. p. 14—18, 1926.

[3] *R. A. Heising, J. C. Schelleng* and *G. C. Southworth* (Proc. Inst. Rad. Eng. 1926) have made a comparison between horizontal and vertical transmitting antennas and come to the conclusion that at distant locations — over, say 500 km — the differences between these antennas disappear. (p. 645, Fig. 26).

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about 1 km the waves, therefore, will be attenuated according to the formula (19) in Chapter III, and the amplitude of the waves will be proportional to:

$$\frac{1}{\sqrt{r}} e^{-\alpha r}, \tag{13}$$

where r is the distance from the transmitter and α the attenuation constant in *Zenneck*'s formulas, see Fig. III. 2 and Table 1. The corresponding value of α is about 1 to 10 km^{-1}, and a wave of this kind will therefore be completely absorbed within a very short distance from the transmitter. With higher aerial the absorption will, no doubt, be considerably less; but the problem has, as far as we are aware, not yet been solved completely. Valuable information about this question may, however, be gained from recent papers by *Eckersley* and others[1].

The attenuation will necessarily also be very great in the case where a ray, returning from the upper atmosphere, is just touching the earth tangentially.

It is hardly possible to calculate the above mentioned losses with any degree of certainty, but they will, as mentioned, be very large and rays, which leave the transmitter in a horizontal direction, cannot therefore be very effective unless the wave length is very small.

If the ray forms an angle of $\psi = 5$ degrees with the earth then this ray will be at a height of one wave length from the surface of the earth at a distance of about 0.25 km from the transmitter — or from a point at which the ray is reflected by the earth. In this case the losses will undoubtedly be very much smaller than for horizontal rays.

In the following we shall therefore in most cases only consider rays with earth angles above 5 degrees.

(b) Attenuation of Earthbound Short Waves. Skip Distances.

The attenuation of the ground waves may, for short waves, be determined by means of *Sommerfeld*'s and *Zenneck*'s theories. The effective distances are here so small that, without committing any serious error, we can consider the earth as flat and make use of the formulas from Chapter III.

In Fig. 1 the field strength is shown as a function of the distance for various wave lengths. The uppermost part of the figure corresponds to ordinary soil ($\sigma = 10^{-13}$ e. m. u.) and the lower part to ocean water ($\sigma = 10^{-11}$ e. m. u.). The values are computed from formula (19) in Chapter III. It appears from Fig. 1 that ground waves for which $\lambda < 40$ m are very strongly attenuated for all kinds of ground and even for $\lambda = 94$ m are strongly attenuated over ordinary soil. In the transmission of short waves the earth bound waves therefore are of minor importance only.

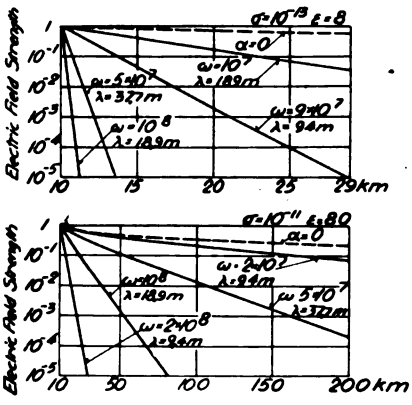

Fig. XI. 1. Electric Field Strength as a Function of the Distance for Short Waves over ordinary Soil ($\sigma = 10^{-13}$ e. m. u.) and over Ocean Water ($\sigma = 10^{-11}$ e. m. u.).

The most characteristic features of short wave transmission are the skipping and the long range ability shown by these waves.

[1] *T. L. Eckersley:* Experim. Wireless, Vol. 4, p. 213. April 1927. (Abstract). *B. Rolf:* Tekn. Tidskrift, p. 200—203. 1925. *R. Mesny:* L'onde électrique. Vol. 6, p. 181—199. 1927.

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The skipping phenomenon is now an undisputed and well known fact, but with regard to the value of the distance the available evidence is rather conflicting. We shall not enter into any detailed investigation of the very numerous publications bearing on this subject, but only quote a few of the most important papers[1].

From these papers we have compiled the information about day skipping contained in Figs. 2 and 3. In Fig. 2 the full line shows the average yearly value of the skip distance for full daylight conditions as given by *Taylor* and *Hulburt* (l. c., Fig. 1). The two crosses mark the values of day skipping during autumn (September—December) as found by *Heising, Schelleng* and *Southworth* (l. c., p. 630, Fig. 17, see Fig. 3 below). The full circles give the values of day skipping in spring (April) according to *Prescott* (l. c.).

The broken line marked P in Fig. 2 shows the skip distances by day in summer as calculated on the basis of the F' atmosphere. For the corresponding state of ionization the values of the refractive index and of the attenuation constant are given in Figs. IX 13 and 14 (in the appendix). The calculations are carried out by means of the formulas and methods given in Chapter X.

It appears from Fig. 2 that the skip values of *Taylor* and *Hulburt* are considerably greater than those found by *Heising, Schelleng* and *Southworth* and by *Prescott*. Considering the fact, that day skipping is generally less in summer than in winter, the broken line agrees fairly well with the values derived from the two last mentioned sources.

The values of the skip distances are generally greater and often much

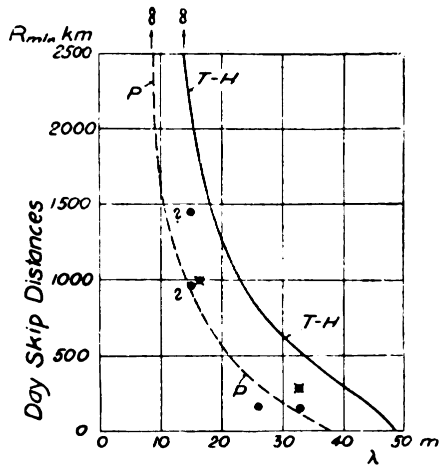

Fig. XI. 2. The Day Skip Distances as a Function of the Wave Length.

The full curve marked *T-H* shows the average yearly value for full daylight conditions according to *Taylor* and *Hulburt*.

The two points marked with crossed black circles correspond to the day skip distance (Sept.—Dec.) taken from the idealized transmission surfaces given by *Heising, Schelleng* and *Southworth* (l. c., Fig. 17 and Fig. 3 below).

The points marked with black circles correspond to the day skip distances given by *Prescott*. (l. c., Figs. 2 and 3).

The two points on the 15 m line marked with a ? refer to the following statement by *Prescott*: »No reports were received on the daylight period within a radius of nine hundred miles from the transmitter, indicating an apparent skip distance of this magnitude. However, observations made by field men previous to the tests of April, gave an apparent skip distance of six hundred miles«[2].

The dotted curve marked P shows the skip distances corresponding to summer ionization at noon and to the atmosphere F', see Figs. IX. 13 and 14 (in the appendix).

The arrows at the top of the figure indicate short wave limit for day transmission assumed respectively by *Taylor* and *Hulburt* and by the writer.

[1] *A. Hoyt Taylor* and *E. O. Hulburt:* Phys. Rev. (II). Vol. 27, p. 189—215, 1926. *R. A. Heising, J. C. Schelleng* and *G. C. Southworth*: Proc. Inst. Rad. Eng. Vol. 14, p. 613—647, 1926. *M. L. Prescott*: »QST«. p. 9—13, Novbr. 1926.

[2] The six hundred miles distance agrees with the night skip distance found by *Prescott* better than does the nine hundred miles.

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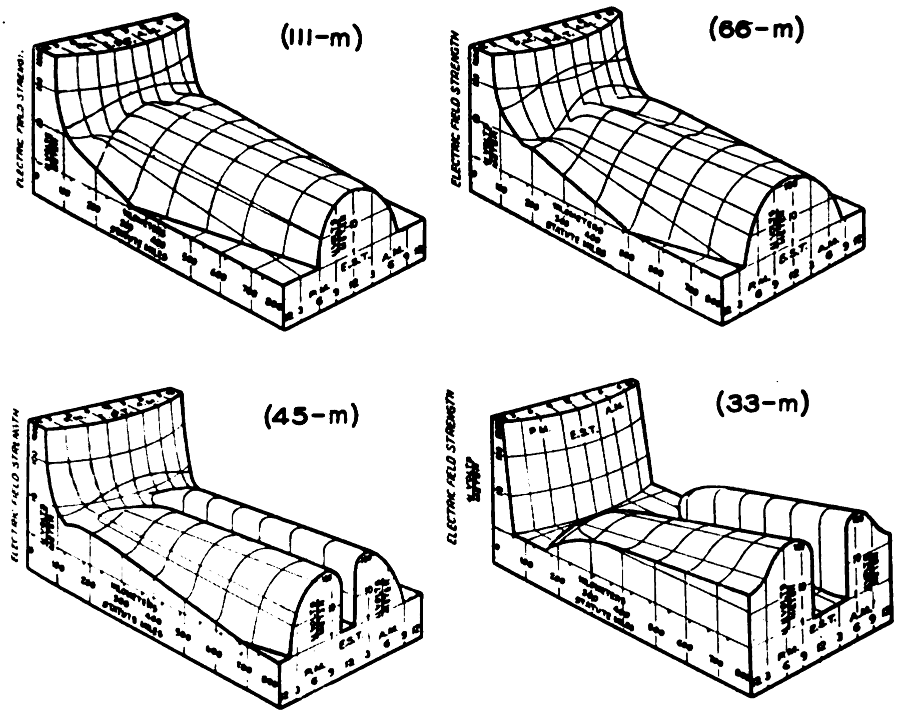
greater by night than by day, but reliable information about the values of the regular night skip distances is rather scanty. We have in Fig 4 shown a few of the results obtained by various authors. The curve marked *T-H* represents skip values which, according to *Taylor* and *Hulburt*, are three times the day



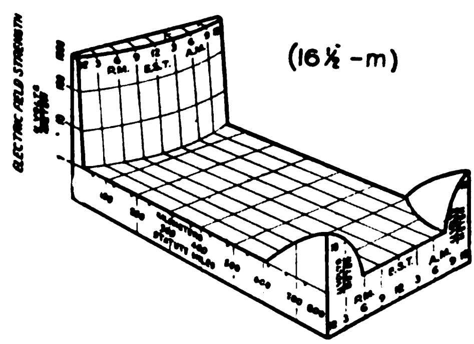


Fig. XI. 3. Idealized Transmission Surfaces. Variation of Field Strength with Time and Distance according to *Heising, Schelleng* and *Southworth*. (The originals for this representation of the very valuable results obtained by these authors have been kindly placed at my disposal by Mr. *Heising*).

skip distances. As these authors say: »the night skip distances are on a rough average three or four times the corresponding day values«. The black circles show the values found by *Prescott*. The circle with the ? on the 15 m line refers to the following remark of *Prescott* (l. c., p. 11): »A skip distance of one thousand miles was indicated for the night period. In the region beyond the day and night skip distances the signal was consistently erratic in its behaviour, ...« In our opinion the 15 m wave will not return regularly at night — 6 to 10 hours after sunset — and this agrees rather well with *Prescott's*

remarks. It is of some interest, however, that when this wave does return at night its skip distance — according to *Prescott* — is about 1650 km.

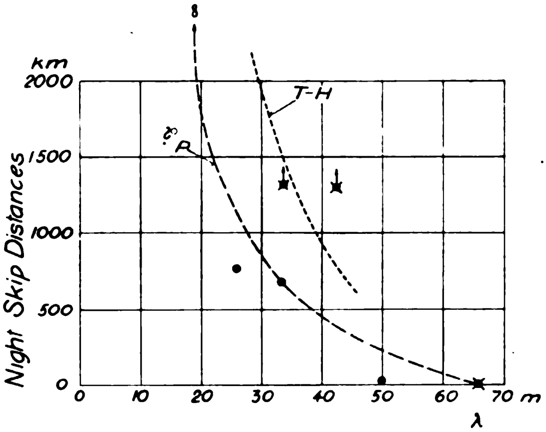


Fig. XI. 4. Night Skip Distances. The curve marked *T-H* indicates the approximate values given by *Taylor* and *Hulburt*. The crosses correspond to the values taken from *Heising, Schelleng* and *Southworth*'s idealized transmission figures. The points marked by black circles represent *Prescott*'s values and the one marked with a ? shows the night skip distance of the 15 m wave according to *Prescott* — when this wave does return.

The crosses represent the values taken from *Heising, Schelleng* and *Southworth*'s idealized transmission figures. The two crosses with vertical arrows indicate that the corresponding skip distances are greater than indicated by the crosses. The curve marked *P* shows the skip distances corresponding to the (n, h)-curve in Fig. IX. 14 (in the appendix).

As far as the present evidence goes we believe the curve *P* represents a fair average.

At sunset or sunrise between the transmitter and receiver the conditions become more complicated — compare the analogous problem for long waves discussed in section 3(e) below.

(c) Short Wave Transmission over Short Distances, i. e. up to 1000 to 2000 km.

In Chapter X we have discussed ray paths as determined by the methods of geometrical optics, we shall therefore here only call the attention to some of the results quoted in that chapter.

Fig. 5. I. shows the value of the refractive index n as a function of the alti-

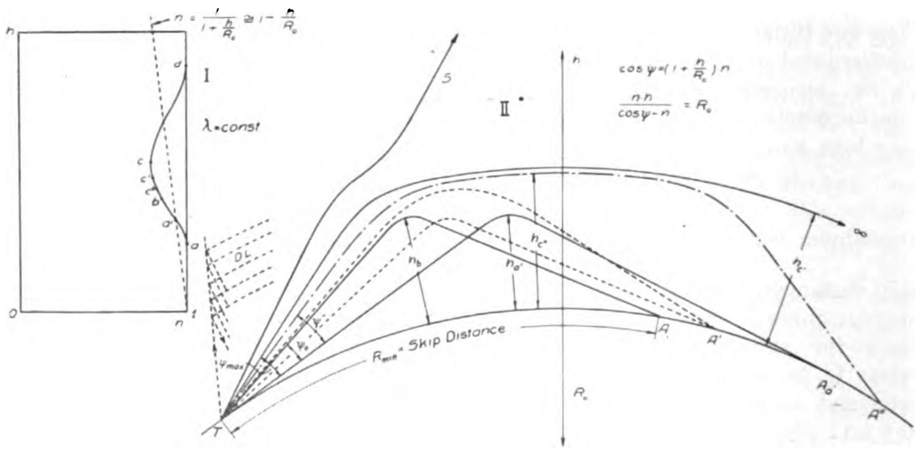


Fig. XI. 5. Diagrammatical Representation of Ray Paths corresponding to the (n, h)-curve $aa' bc'' c' c$ in part I.

tude h. The top point of the ray path leaving the transmitter T tangentially will reach an altitude $h_{a'}$, corresponding to a value of n determined by

$$n_{a'} = \frac{\cos\psi}{1+\frac{h_{a'}}{R_0}} = \frac{1}{1+\frac{h_{a'}}{R_0}} \cong 1 - \frac{h_{a'}}{R_0}. \tag{14}$$

This ray vill just come down to the earth again and touch it tangentially at the point $A_{a'}$. In most cases this ray will not be very effective in transmission, because, as mentioned above, it will be very strongly attenuated on account of earth losses both at the transmitter and the receiver.

The ray TA_b leaves the transmitter at such an earth angle ψ_0 that the distance TA_b has its smallest value R_{min} = the skip distance. The altitude of its highest point, h_b, will be somewhat greater than the altitude of the point of inflexion b of the (n, h)-curve.

The ray $T\infty$ leaves at such an earth angle ψ_{max} that the radius of curvature at the top of its path is equal to $(R_0 + h)$. The corresponding point c' of the (n, h)-curve is just a little lower than the point c in which the value of n is a minimum. This ray will never come back to the earth but will approach the circle with radius $(R_0 + h_{c'})$.

Any ray S leaving at an earth angle greater than ψ_{max} will penetrate the ionized part of the atmosphere and get lost in free space.

Any point A' between A_b and $A_{a'}$ will receive two rays, one having an earth angle smaller, the other greater, than ψ_0. Any point A'' outside the point $A_{a'}$ will only receive one ray corresponding to an earth angle a little smaller than ψ_{max} and to a point c'' between the points b and c' on the n-curve.

For longer waves the rays having an earth angle greater than ψ_0 may be so strongly attenuated along the curved path in the ionized part of the atmosphere that they are practically obliterated. Even the ray corresponding to ψ_0 may be so strongly attenuated that the field strength at A_b is very small. This attenuation along the curved path of the ray may be calculated approximately from the formulas given in Chapter X sect. 5. For short waves it is generally so small that it will be of minor importance with regard to the rays received between A_b and $A_{a'}$. For long waves this attenuation will be considered later.

The field strength at a receiving point A' will therefore mainly depend on the geometrical distribution of the radiated energy. If the shape of the lower part $aa'bc''c'c$ of (n, h)-curve was given we might calculate both the attenuation along the curved part of the paths and the effect of the geometrial distribution of the radiated energy. The result would depend very much on the exact shape of the lower part of the (n, h)-curve and unfortunately we know very little about the form of this part of the curve.

For Figs. IX 6 to 20 (in the appendix) we have based our calculations on the assumption that the ionization is caused by one single radiation with a definite absorption constant, corresponding to a monochromatic radiation. However, the ionizing ultra-violet radiation is certainly more complex, but we know too little about it to make any specific assumptions. We have therefore, quite arbitrarily, assumed the ionizing radiation to be monochromatic, but in order to take some account of the inhomogeneity which no doubt exists in this radiation we have made the lower boundary of the ionization fall off less rapidly than it would have done if the radiation had been purely monochromatic, see Chapter IX, Table 7.

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For this assumed lower boundary of the ionization we may, as mentioned above, calculate the distribution of the field strength between A_b and $A_{a'}$ for the wave length considered. We shall not enter into the details of these calculations but only give a few results for a 37.7 m wave ($\omega = 5 \cdot 10^7$) (Γ taken to be twice the value determined by Chapt. X, (83)):

ψ	Distance	Field Strength	Remarks
5°	1470 km	$181 \cdot 10^{-6}$ volts m^{-1}	The field strength
10	1020 »	169· » » »	corresponds to a
20	580 »	157· » » »	radiated energy of
30	390 »	125· » » »	1 kW.
35	320 »	105· » » »	Daytime.
			Atmosphere *F*.

A comparison between these values of the field strength and those given by *Heising, Schelleng* and *Southworth* (l. c., Fig. 14) shows that our values correspond fairly well with the highest values obtained by these authors, while the average values found by them are about 10 times smaller. As we in our calculations have taken no account of losses occurring in the earth or in other obstacles in the way of the waves, our values should be considerably higher than the actual mean values.

As stated before it is hardly possible to calculate these losses with any accuracy, but they must necessarily be very heavy and especially for the rays having small earth angles. (We have for this reason left out rays for which $\psi < 5^0$. *Very* short waves may, however, be effective for smaller values of ψ).

All in all we believe the agreement to be as close as could be expected.

As pointed out before and as shown in Figs. 2 and 4 the skip distances calculated from the values of n given in Fig. IX. 14 (in the appendix) also agree fairly well with experience. Therefore at present we see no reason to alter the distribution of the intensity of ionization or its altitude. But these features may be altered somewhat without altering the main lines in the present theory should further evidence make it desirable.

If the skip distance was accurately known as a function of the wave length it would be possible also to determine accurately the altitude and distribution of the ionization. And from the distribution of ionization thus found it would be possible to determine the constitution of the ionizing radiation as far as the distribution of the radiation energy over various values of the absorption constant is concerned.

We believe, however, that it would be a little premature to try to do this at present since the evidence concerning the regular values of the skip distances is still rather conflicting.

(d). Short Wave Transmission over Long Distances, *i. e.* Distances over about 8000 km.

Before taking up the discussion of this problem it is necessary to consider some of the main features connected with reflection of the waves at the surface of the earth.

For reflection at a plane surface the loss is always greater for short than

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for long waves, see f. inst. Figs. VIII. 11, 13, 15 and 17. The loss is always greater for waves in which the electric field is parallel to the plane of incidence than for the same waves with the electric field perpendicular to this plane, see Table 8 below. It appears from this table and from an inspection of the above mentioned figures that the reflection losses are very heavy for short waves with their electric field parallel to the plane of incidence and with the small earth angles which are of special importance for the problem at hand. In some cases even less than one thousandth part of the incident energy is reflected.

Table 8. Reflection Losses for a 18.9 m Wave ($\omega = 10^8$).

Character of the soil	Angle of incidence	The electric field parallel to the plane of incidence Case I		perpendicular to the plane of incidence Case II	
		$\left\|\frac{a_r}{a_i}\right\|$	$\left\|\frac{a_r}{a_i}\right\|^2$	$\left\|\frac{a_r}{a_i}\right\|$	$\left\|\frac{a_r}{a_i}\right\|^2$
Ocean Water..	85°	0.62	0.385	0.985	0.97
Fresh Water..	84°	0.02	0.0004	0.98	0.96
Wet soil......	72°	0 15	0.0225	0.83	0.66
Dry soil......	63°	0.03	0.0009	0.60	0.36

It will be shown later that the direction of the electric field in the wave front of the incident ray is quite arbitrary. We may, therefore, without committing any serious error, assume the mean loss of energy at a reflection to be about 50 per cent for the 18.9 m wave, a little less for longer waves, a little more for shorter.

In estimating this loss we have assumed the reflecting surface to be absolutely plane, horizontal and homogeneous. Actually this will very seldom be the case. And even for plane, horizontal surfaces the losses will often be much greater. As mentioned above this will be the case if the incident and reflected rays have very small earth angles. Thus we have seen that short waves moving tangentially along the surface of the earth will suffer so heavy an attenuation as to be practically obliterated. We therefore only consider rays which have an earth angle above 5°. But even for such rays vegetation, trees and other obstacles may cause very heavy losses also for waves in which the electric field is at a right angle to the plane of incidence. We are hardly overestimating this loss by assuming its mean value to be another 50 per cent and we therefore assume the mean value of the energy in the reflected ray to be 25 per cent of that in the incident ray at reflection from a plane horizontal surface. For waves shorter than 18.9 m this loss will be somewhat greater and for longer waves somewhat smaller.

But the reflecting surface will very seldom be plane and horizontal and the reflected energy will therefore in most cases be scattered, at least partly, in such directions that it is lost as far as long distance transmission is concerned.

If, f. inst., the reflecting surface is plane but makes an uphill angle β with the horizon the reflected ray will have an earth angle of $\psi + 2\beta$, ψ being the

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earth angle of the incident ray. If ψ is only a little smaller than ψ_{max} even very small values of β will cause the reflected energy to get lost in space. In most cases, however, the reflecting surface will not be plane and the scattering therefore more complicated.

In Fig. 6 we have supposed the surface *ABCDE* to have triangular hills or ripples at right angles to the direction of propagation, the hill sides having an earth angle equal to β. In this case all the reflected rays will have earth angles greater than or at least equal to 2β. Any incident ray, as 3, with an earth angle greater than β will have two reflected rays, 3′ and 3″.

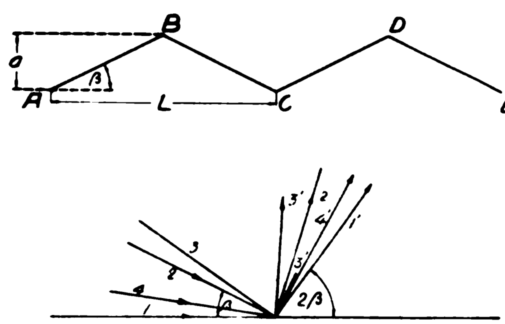

Fig. XI. 6. Reflection from a Rippled Surface. 1, 2, 3 and 4 are the incident 1′, 2′, 3′, 3″ and 4′ the reflected rays.

In Fig. 7 we have shown the reflected ray for three different incident rays 1, 2 and 3. The two first having two and the last three reflected rays.

In the general case where the surface is more or less irregular an incident ray — as S_i in Fig. 8 — will not give one single or a few reflected rays but will result in the reflected energy being more or less evenly scattered within certain angular limits. If the irregularities of the surface are small in comparison to the wave length — which generally will be the case with very long waves — then the reflected energy will very nearly be concentrated in the single ray S_r having the same earth angle ψ as the incident ray. When the wave length decreases so far that the dimensions of the irregularities become of the same order of magnitude as the wave length the reflected energy will be scattered according to some distribution curve, as for example III in Fig. 8. As the wave length decreases further the energy fraction reflected in a direction not more than a few degrees different from the direction of the regularly reflected ray S_r will also decrease as indicated by the distribution curves II and I.

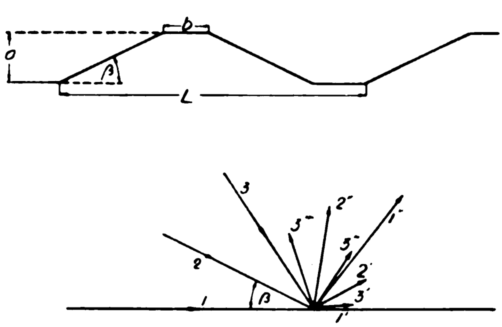

Fig. XI. 7. Reflection from a Wave Shaped Surface.

If we assume all rays having an earth angle smaller than $\psi' = \angle LAb$ to be lost on account of their great attenuation in passing along the surface of the earth and all reflected rays having an earth angle greater than ψ_{max} to be lost by escaping to free space then only that part of the reflected energy for which $\psi' \leqq \psi \leqq \psi_{max}$ will have any possibility of contributing to the field strength at the receiver, while the rest of the reflected energy will be irrevocably lost.

These scattering losses will for short waves in many cases be very great not alone by reflection at irregular surfaces of land but even by reflection at

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the surface of the sea, — apart from the rather rare cases where this surface is completely even without surges, waves or ripples.

Besides this scattering in the plane of incidence there will also be lateral scattering and most of the thus scattered energy is lost as far as the transmission problem is concerned.

It is very difficult to estimate the average value of the scattering loss. But if we, for the 18.9 m wave, assume the value of $(\psi_{max} - \psi')$ to be equal to 20^0 we can hardly take the value of this loss to be less than about 60 per cent. For longer waves this loss will be considerably less both because the scattering itself will be smaller and because the angle $(\psi_{max} - \psi')$ will be greater.

For an 18.9 m wave we accordingly assume that on an average only 10 per cent of the incident energy will be reflected in such a manner that it may possibly contribute to the field strength at the point of reception.

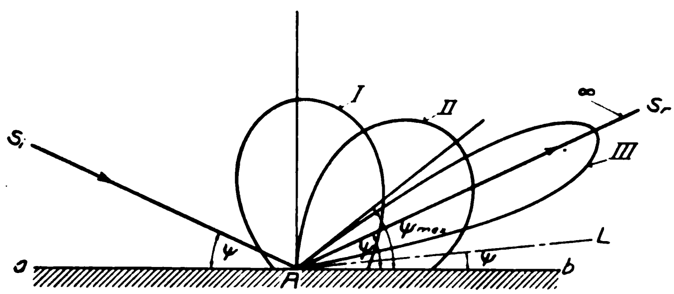

Fig. XI. 8. Diagrammatical Representation of the Scattering by Reflection at an irregular Surface. S_i is the incident ray, S_r the reflected ray for $\lambda \rightarrow \infty$. The curves III, II and I show the distributions of the reflected energy for waves of decreasing wave length. (The angle LAb should be ψ' instead of ψ).

This average value of the reflection efficiency may be rather erroneous, but at all events there can be no doubt whatever that its value must be increasing rapidly with increasing wave length and this fact is of even greater importance in the following considerations than is its absolute value.

We can now discuss the problem of long distance transmission by means of short waves and we shall do so by making use of schematical ray paths as shown in Fig. 9. A is the transmitting station and R the receiver. For the sake of simplicity we have assumed the earth to be flat.

We shall first consider a transmission by means of a great number of steps as in Fig. 9. I. If the surface is perfectly plane and horizontal and if the electrical properties of the air depend solely upon the altitude then any ray — f. inst. the ray marked 1 — having a »step« AA'_1 which is a submultiple of the total distance $AR = r$ will reach the receiver, more or less attenuated. If s is the skip distance the number p of possible paths in this case is the greatest integral number $\leqq \frac{r}{s}$. This case is, however, very improbable.

We shall next discuss the actual case where the surface at the points of reflection is more or less irregular. In this case the conditions are much more complicated. Each ray — f. inst. the ray marked 2 — leaving the transmitter A will at the first point of reflection A''_1 lose a part a of its energy by absorption in the ground, another part s will be lost by scattering while a third

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part r is reflected in such a direction that part of it may reach the receiver. We have only shown one of the reflected rays leaving the point A_1'' and this ray is again reflected at A_2'', where the same kind of losses occur and where a beam of possibly useful rays is again reflected. Of these we have again only shown one reaching the earth again at A_3' and so on. We call this ›short-step‹ transmission.

Besides the losses already treated there is also some loss along the curved path at the top of the rays. This loss for each step is denoted by t. If by

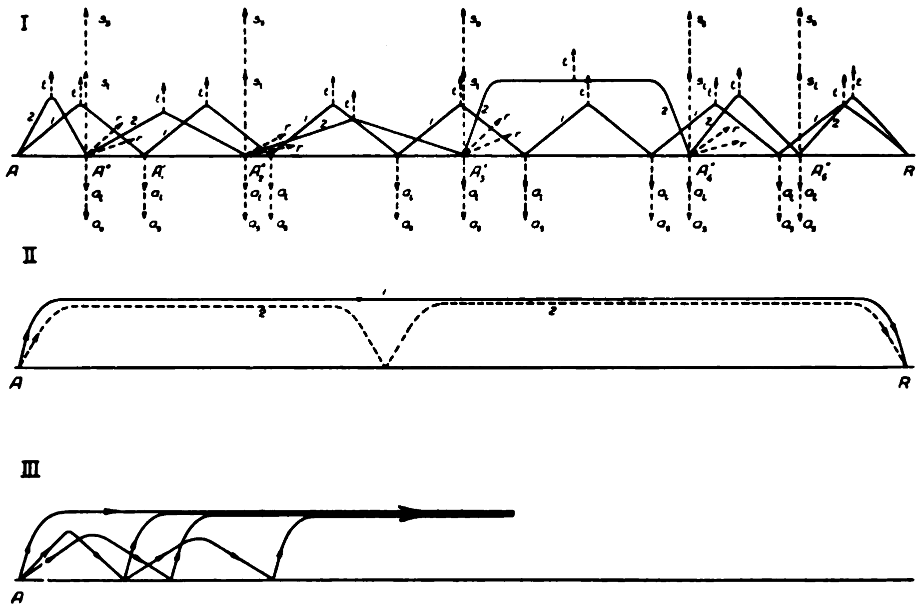

Fig. XI. 9. Various Possible Ray Paths in Long Distance Transmission. I represents ›Short-step‹ Transmission, II and III ›Long-step‹ Transmission.

index s we refer to a short wave and by index l to a somewhat longer wave, we have as mentioned before:

$$s_s > s_l \quad \text{and} \quad a_s > a_l. \tag{15}$$

Since the values of n for the short waves and short steps here considered are almost the same at the paths of the ›short‹ and the ›long‹ waves the values of the attenuation constant γ_0 will also be almost the same, since in accordance with formula (42) in Chapter VI we have:

$$\sigma = \frac{v}{4\pi c^2} \cdot \Delta\varepsilon = \frac{v}{4\pi c^2} \cdot (1 - n^2). \tag{16}$$

The losses t_s and t_l will therefore in general be very nearly equal.

A general feature of all the ray steps in Fig. 9. I — with the exception of the step $A_3' \to A_4''$ — is that only a comparatively small part of the total path is lying in the upper, highly ionized part of the air. The consequence is that in order to travel very long distances the ray must make a great number of ›steps‹. For a distance of 20 000 km this number must be at least 10 to 15, and in order to go round the whole earth about 20 to 30.

In Fig. 9. II we have shown another manner of transmission in which a

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ray, 1, goes directly from A to R, or at all events only makes a very limited number of steps, as indicated by ray 2, which makes two steps.

The question is now: in which of the two possible manners does the transmission actually take place? As stated above it cannot be in the short-step manner indicated in I, because the reflection losses are too great and the reflected energy too erratic to allow any appreciable amount of energy to be regularly transmitted in this way, when the number of steps is great. If we assume the efficiency of the reflection to have the value estimated above, namely 0.1, then the energy would be reduced 10^{10} to 10^{15} times in being transmitted about 20 000 km and 10^{20} to 10^{30} times in being transmitted around the whole earth. In these calculations no account is taken neither of the losses in the ionized part of the atmosphere nor of the geometrical spreading out of the energy.

The transmission therefore must take place in the long-step manner indicated in II.

We may prove this also in another way:

As pointed out before, the reflection losses decrease considerably with increasing wave length. If, therefore, transmissions take place according to I the intensity at the receiver must necessarily be greater for the longer waves than for the short ones. But this is contrary to experience: Waves from 15 m up to about 20 m may be effectively transmitted over very great distances by day while waves longer than about 24 m do not come through.

On the other hand this fact is in complete agreement with the long-step transmission. For the long steps here taken, the top of the ray path will not be at the height for which the refractive index has the same value for the two waves considered but must be at such a height, that the radius of curvature is very nearly equal to $(R_0 + h)$. But the altitudes at which this takes place generally vary very little for moderate alterations in the wave length. The electron density at the top of the path therefore will be almost the same for an 18 m and for a 24 m wave, and the attenuation constant for the long wave will be about $(\frac{4}{3})^2 = \frac{16}{9}$ times that for the short wave. According to the preceeding this is quite sufficient to explain the difference in reaching ability of the two waves.

As far as we can see there is only one other possibility left, namely that the top of the ray path comes up to very high altitudes, 500 to 1000 km or even more, in which case the steps may be so long that their number even for the longest distances will be comparatively small. This hypothesis, assuming very great altitudes of even the lower boundary of the ionized region, has been discussed by some authors[1], but for geophysical reasons we are led to consider such an alternative as very improbable, and its untenability is conclusively proved by the experimental fact that the top of the ray path is below about 200 km, a fact which will be discussed later.

In our opinion, therefore, the transmission of the short waves must take place in the long-step manner shown in Fig. 9, II.

One difficulty, however, remains. It appears from Fig. 5 that the whole energy radiated within the angle ψ'' returns to the earth between the points A_b and A'', while only that small part of the radiated energy which falls within the angle $(\psi_{max} - \psi'')$ is left to be distributed over the whole area outside the point A''. The value of this angle $(\psi_{max} - \psi'')$ will depend very much

[1] See f. inst. *Baker* and *Rice* l. c.

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on the shape of the (n, h)-curve in the vicinity of the point c'. But rather improbable assumptions would have to be made in order to obtain, in all cases, a considerable value of this angle.

At first sight this difficulty may, perhaps, look very serious, but it is not at all as grave as it looks.

It will of course be very important that as much as possible of the radiated energy is concentrated within this angle — and an effective beam system concentrating the greater part of the radiation within this small angle will no doubt prove very effective. — But the natural energy conditions are not at all as unfavourable as they appear. In order to illustrate this we have prepared the schematical drawing shown in Fig. 10 and we make the following assumptions which, no doubt, are not at all justified but which may still serve our present purpose: There is no absorption neither by reflection nor along the

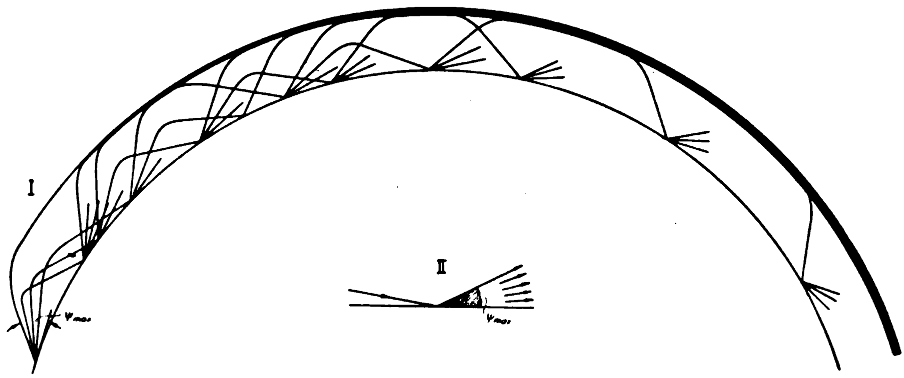

Fig. XI. 10. Diagrammatical Representation of the Ideal Case of Long Distance Energy Concentration.

paths of the rays and the scattering at reflection takes place in such a manner that the whole energy after reflection is concentrated within the earth angle ψ_{max} corresponding to the ray having an infinitely long transmission distance, see part II of Fig. 10.

It is easily seen that in this case all subsequent reflections will contribute, more or less, to that part of the radiation energy which corresponds to earth angles very close to ψ_{max}, and as the transmission distance increases the energy would be more and more concentrated within the earth angles ψ_{max} and $(\psi_{max} - \Delta\psi)$.

This is, of course, a highly idealized case. As mentioned above the reflection losses are rather great and a ray having made a number of reflections will therefore only be very little effective in contributing to the accumulation of the long distance radiation energy. But still it is evident that for the long distance transmission not only the energy radiated within the angle $(\psi_{max} - \psi'')$ in Fig. 5 will be useful but also an appreciable part of the energy radiated within the angle ψ'' will ultimately contribute to this long distance radiation.

The schematical ray paths in Fig. 9. II must therefore be modified in so far as the whole energy carried by the long-distance rays is generally not delivered directly from the transmitter, but some part of it has made one or more introductory steps. Part III of Fig. 9 shows a schematical representation of this view of long distance transmission of short waves.

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(e) Altitude of the Maximum Ionization and of the Maximum Electron Density.

The altitude of the maximum ionization will be greater by night than by day and greater on a winter day than on a summer day, but the value of the maximum ionization will be greatest on a summer day, somewhat less a winter day and considerably less by night. The altitude h_{Di} of the maximum

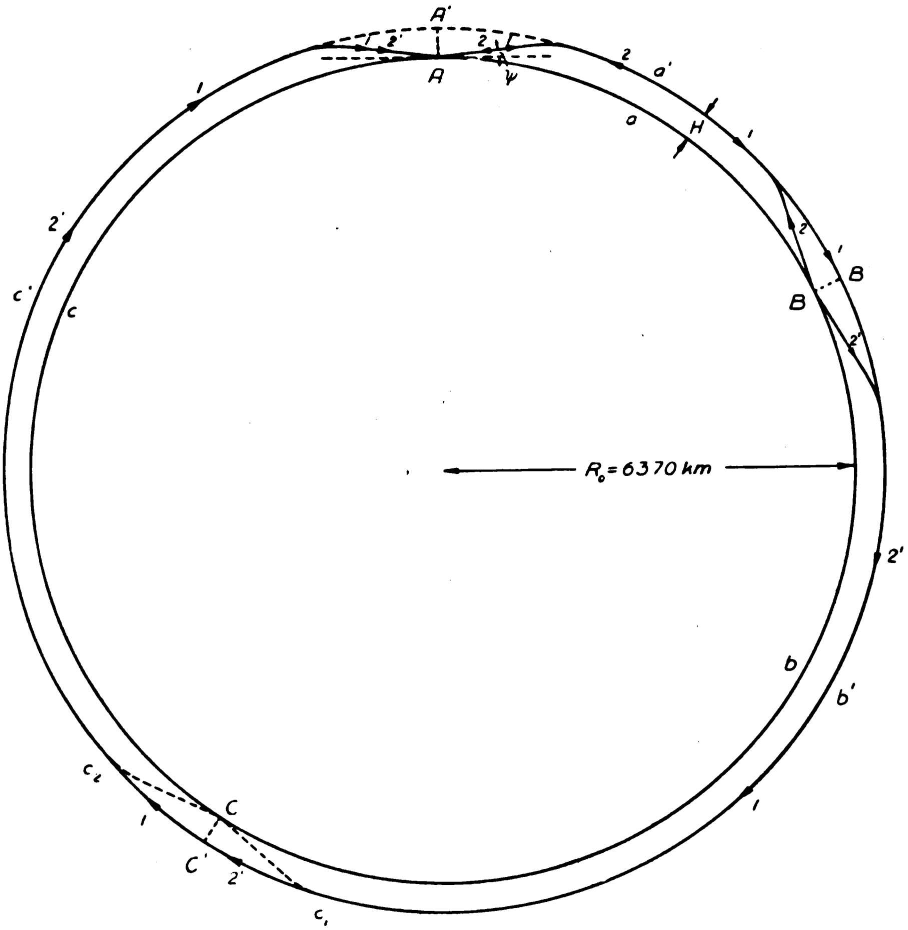

Fig. XI. 11. Various Ray Paths. Ray 1 is transmitted from the station A and comes back to this station after having encircled the globe. The two rays, 2 and 2', transmitted in opposite directions from the station B, will both arrive at the station A.

ion density is a little and the altitude h_{De} of the maximum electron density considerably greater than the altitude h_m of maximum ionization. These conditions are for the atmosphere F' shown in Figs. IX 6, 7 and 8 (Figs. 6 and 7 in the appendix).

A series of investigations has been carried out aiming at the determination of the lower boundary of the ionization, but these investigations have been made with longer waves and we shall therefore discuss these papers later. It should be borne in mind, however, that strictly speaking, the different experi-

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ments can only give us the values of n and possibly of γ, and it is only for the short waves that the n-curve has practically the same shape as the electron density curve. As far as we are aware there has been no direct experimental evidence regarding the altitudes of maximum ionization and of maximum ion and electron density, and such evidence can presumably only be obtained by means of short waves and only for the electron density. There is, however, some indirect evidence with regard to this question, which we shall now consider. From reliable values of skip distances for short waves it would be possible to estimate the altitude and shape of the lower boundary of the n-curve and therefore also the lower boundary of the electron density.

But, as stated before, the available evidence concerning skip distances is too conflicting to be of great service. All in all we feel justified in saying that the ionization curves shown in Figs. IX 6 and 7 (in the appendix) may possibly agree with reliable skip evidence — when such evidence is available — or else they may by small alterations in shape and altitude be made to do so.

There is another possibility of gaining some information with regard to this question. The German Telegraph Administration and the *Telefunken* Company have made a series of interesting experiments in order to determine the time τ occupied by a short-wave signal to encircle the globe[1], see Fig. 11. They also measured the time interval τ between the arrival of the direct signal (2) from an American short-wave station (B) and the arrival of the indirect or echo signal (2′) coming the other way round the globe. The results are collected in Table 9. The values of τ and of A_0 have been taken directly from the

Table 9. Altitudes of the Tops of the Ray Paths.

Great Circle Distances A_0	τ	$A_1 = c\tau$	$L = A_1 - A_0$	H	λ	H′ $\psi=0$	H′ $\psi=5^0$	H′ $\psi=10^0$
	sec.	km	km	km	m	km	km	km
BbCcA — BaA* =	0.0955	28 630	1290	ab. 305	16.175	150	134	98
27 340 km	0.0945	28 330	990	» 234	22.0	116	101	64
AaBbCcA* = 40 000 km	0.1407	42 180	2180	» 350	15.0	174	157	120

* See fig. 11.

figures published by *Wagner*[2]. A_0 is the difference between the distances traversed by the echo signal and by the direct signal and measured along a great circle at the surface of the earth. The above mentioned authors assume the signal velocity to be equal to the velocity of light and this velocity to be $c = 299\,800$ km per second and therefore take the difference in path-length to be equal to $A_1 = c\tau$. The length of the ray path is accordingly $L = (A_1 - A_0)$ greater than the corresponding great circle distance A_0 and the altitude H of the path is determined by

[1] *E. Quäck:* Jahrb. d. drahtl. Tel. Bd. 28, p. 177—178. (Decbr.) 1926.

[2] *K. W. Wagner:* (a) Sitzungsber. d. Berliner Akad. Bd. 33, p. 475 (16. Decbr.) 1926. (b) »Forschungen und Fortschritte«, 3. Jahrg. No. 3, p. 20—21. (20. Jan.) 1927 (c) E. N. T. Bd. 4, p. 74—76. 1927. There seems to be some errors in the figures given by *Quäck*, and as *Wagner*'s papers (b) and (c) are of a later date and more complete we have made use of his figures only.

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$$H = \frac{L}{A_0} \cdot R_0 = \frac{A_1 - A_0}{A_0} \cdot R_0 = \frac{A_1 - A_0}{A_0} \cdot 6370, \tag{17}$$

R_0 being the radius of the earth.

The values of H thus calculated are given in Table 9 under the heading H.

According to our view of the transmission problem it will be necessary, however, to make an important change in carrying out these calculations[1]. We have proved in Chapter X, formula (68), that the signal- or group-velocity u is given by

$$u = \frac{nc}{\left(\frac{dt}{dt'}\right)}, \tag{18}$$

and that for $\nu \ll \omega$ is $\frac{dt}{dt'} \simeq 1$. In this case we have

$$u = nc, \tag{19}$$

and it is this velocity u with which a signal will travel in the upper part of the path.

According to this we ought to calculate the path length A_1' as

$$A_1' = nc\tau \tag{20}$$

instead of as $A_1 = c\tau$.

The difference L' between path length and corresponding great circle distance is then

$$L' = A_1' - A_0 = nc\tau - A_0, \tag{21}$$

and the altitude H' of the path is determined by

$$H' = \frac{L'}{A_0} \cdot R_0 = \frac{nc\tau - A_0}{A_0} \cdot R_0. \tag{22}$$

The next question is: What will be the value of the refractive index n at the upper part of the path? According to formula (10) in Chapter X we have

$$n = \frac{R_0 \cos\psi}{R_0 + H'} = \frac{\cos\psi}{1 + \frac{H'}{R_0}}, \tag{23}$$

where ψ is the earth angle of the ray in leaving the transmitting station, see Fig. 11.

It appears from (23) that at all events

$$n < \frac{1}{1 + \frac{H'}{R_0}} \simeq 1 - \frac{H'}{R_0}. \tag{24}$$

Hence from (22) and (24):

$$H' < \frac{c\tau - A_0}{c\tau + A_0} \cdot R_0 = \frac{L}{c\tau + A_0} \cdot R_0 \simeq \frac{L}{2A_0} \cdot R_0. \tag{25}$$

But it must be assumed that the earth angle ψ is greater than zero and in this case we get instead of (25) the following equation:

[1] *P. O. Pedersen:* »Ingeniøren« No. 9, p. 101—104. (Copenhagen; February 1927). The following treatment is practically taken from this paper. See also: *G. W. O. Howe:* Experim. Wireless, Vol. 4, p. 259—260. (May) 1927.

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$$H' = \frac{c\tau \cos\psi - A_0}{c\tau \cos\psi + A_0} \cdot R_0. \tag{26}$$

We have in Table 9 given the values of H' for $\psi = 0^0$, 5^0 and 10^0.

For the state of ionization assumed by us the maximum density of electrons by day is at an altitude of about 130—135 km and at about 150—155 km by night. The values of the angle ψ must presumably be between 5^0 and 10^0 for the 15 and 16.175 m waves and for the ionization here assumed. We therefore believe these figures agree as well as could be expected with the values of H' in Table 9.

It must be remembered that the values of H' calculated here correspond to the altitudes at which $-\frac{dn}{dh} = \frac{n}{6370 + h}$, *i. e.* the tangent to the (n, h)-curve is very nearly vertical at this altitude which will therefore only be a little less than the altitude corresponding to the minimum value of n. We may therefore, without committing any serious error, take H' as the altitude of the maximum of electron density. It will be quite different, however, in the experiments with longer waves; these longer waves will penetrate more or less deeply only into the lower boundary of the ionization. We shall discuss the question later.

In accordance with *K. W. Wagner* we have assumed the ray path in question to be a circle with a radius of $(R_0 + H)$, see Fig. 11. We have proved above, that this assumption must be substantially justified at least for the short waves, 15 m and 16.176 m; but we have also proved that the ray may, at some point or other, bend down to the surface of the earth and there be reflected again, as indicated by C in Fig. 11. If the earth angle is between 5^0 and 10^0 it will very nearly take the same time for the signal to transverse the two ray paths C_1CC_2 and $C_1C'C_2$ and it will therefore make little difference whether the ray bends down once or twice.

For the 22 m wave the case is somewhat more doubtful; here the ray may possibly make a number of steps and thereby partly evade the highly ionized part of the atmosphere. It is therefore only to be expected that the values of H' found for this wave are less than the values for the shorter waves. The greater weight must, no doubt, be placed on the values found for the shorter waves.

The value of the altitude of maximum ionization and of maximum ion and electron density here assumed thus falls within the limits of the experimental evidence; but this evidence is so scanty and uncertain that these limits are rather wide. When more precise information is available it may possibly be necessary to alter the assumed state of ionization somewhat.

It appears from Table 9 that the altitude of the top of the long distance rays, and accordingly also the altitude of the maximum electron density by night, must be less than 200 km. On the other hand the altitude of the top of the ray path in long distance day transmission for very short waves, *i. e.* for $\lambda < 18{,}9$ m must be at least 115 km. Otherwise the attenuation would be too high.

This is easily seen in the following manner: For the atmosphere F and for an altitude h_D of maximum electron density by day equal to about 130 km (see Figs. IX 6 and 7, in the appendix) the propagation constant for the $\omega = 1.25 \cdot 10^8$ ($\lambda = 15$ m) wave has a value of about $\gamma_0 = 1 \cdot 10^{-3}$. If we reduce h_D, the value of the short-wave limit for day transmission being kept constant,

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the value of γ_0 will increase as shown in the second line in the following table.

$h_D =$	130	120	110	100	90 km.	
$\gamma'_0(h_D) =$	$1\cdot10^{-3}$	$2.4\cdot10^{-3}$	$8\cdot10^{-3}$	$3.2\cdot10^{-2}$	$0.14\ km^{-1}\cdot$	(F')
$\gamma''_0(h_D) =$	$3.5\cdot10^{-4}$	$1.3\cdot10^{-3}$	$6\cdot10^{-3}$	$2.8\cdot10^{-2}$	0.14 »	(F'')

Experience with 15 to 16 m waves shows that the value of $\gamma_0(h_D)$ cannot be appreciably greater than $1\cdot10^{-3}$— this value being perhaps even a little too great —. With the free paths assumed for the atmosphere F' the altitude of maximum daytime electron density in summer cannot be less than about 130 km.

If we assumed either that there was no helium at all in the upper atmosphere or that the effective cross-sectional area of the helium molecules, for the slow electrons here considered, is practically equal to zero, the collision number for the electrons would be represented by the straight line marked v''_{el} in Fig. IX. 1 (in the appendix) and the attenuation constant would attain the values given in the last line in the above table. Even in this very improbable case the value of h_D could not be less than about 120 km.

We are therefore led to the following conclusions:

(ζ). The altitudes of maximum electron density must be between 200 km and 120 km.

(η). The altitudes of maximum electron density here assumed fall within the limits indicated by present short-wave experience.

Some prominent authors have, even quite recently, been of the opinion that the ionizing radiation from the sun penetrates down to very low altitudes and causes a high conductivity there. Thus *J. H. Dellinger, L. E. Whittemore* and *S. Kruse*[1] assume that the atmosphere by day is ionized through the whole stratosphere down to the top of the troposphere at an altitude of about 10 km.[2] *S. Chapman*[3] and *T. L. Eckersley*[4] assume that in the daytime the lower boundary of the ionized layer is at an altitude of 40 to 50 km. *G. Angenheister*[5] assumes that the currents which cause the solar and lunar diurnal variations of the terrestrial magnetism are to be found at an altitude of about 50 km.

We have previously given our reasons for discarding the hypothesis of such low altitudes of the maximum ion or electron density in the »radio layer« and shall here only call attention to Table 4 p. 73 from which it appears that the energy of the ultra-violet end of the solar spectrum is far too small to give the necessary ionization if the height of the maximum ionization caused by the sun is about 50 km and if the short-wave limit for night transmission is about $\lambda_{RN\,min} = 18.9$ m, the value accepted here. We must, from all points of view, consider such low altitudes of the effective radio layer as very improbable.

About the same may be said about the conductive layer required by the theories of terrestrial magnetism.

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[1] *J. H. Dellinger, L. E. Whittemore* and *S. Kruse:* A Study of Radio Signal Fading. Sc. Pap. Bur. Stand No. 476. 1923.

[2] l. c., Fig. 9.

[3] *S. Chapman:* Q. J. Roy. Met. Soc. Vol. 52, p. 225—236. 1926.

[4] *T. L. Eckersley:* l. c.

[5] Handb. der Physik. Bd. XV., p. 306. 1927.

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S. Chapman has shown that the total conductivity for these currents in the daytime must be about 25.10^{-6} (e. m. u., cm)[1], and if we assume one fifth of this conductivity to be due to a layer of air 10 km thick, then the corresponding conductivity will be

$$\sigma = \frac{5 \cdot 10^{-6}}{10 \cdot 10^{5}} = 5 \cdot 10^{-12}. \qquad \text{(e. m. u.)} \qquad (27)$$

If we, in accordance with *G. Angenheister*, assume the middle of the layer to be at an altitude of 50 km, the conductivities caused by one electron and one ion per cubic cm respectively will, according to Chapt. VI. Fig. 1, be equal to

$$\sigma_{0\,\mathrm{el}} \cong 1.5 \cdot 10^{-21} \quad \text{and} \quad \sigma_{0\,\mathrm{ion}} \cong 1.5 \cdot 10^{-24}, \qquad \text{(e. m. u.)} \qquad (28)$$

and the constant of recombination will be

$$\alpha \cong 1 \cdot 10^{-9}. \qquad (29)$$

If we suppose that the conductivity is due solely to ions, their density n_0 is determined by

$$2n_0 = \frac{5 \cdot 10^{-12}}{1.5 \cdot 10^{-24}} = 3.3 \cdot 10^{12} \text{ ions per c c.} \qquad (30)$$

The corresponding ionization I is determined by

$$1.65 \cdot 10^{12} = \sqrt{\frac{I}{10^{-9}}} \quad \text{or} \quad I = 2.7 \cdot 10^{15} \text{ pairs of ions per c c per sec.} \qquad (31)$$

The total number of pairs of ions liberated per second within a sunbeam column of one sq cm cross section is therefore $2.7 \cdot 10^{15} \cdot 10^{6} = 2.7 \cdot 10^{21}$. But according to Table 4. p. 73 this number is at least $1.35 \cdot 10^{9}$ times too great.

If we calculate the number of free electrons according to formula (38a) in Chapt. V, assuming τ to have the value $4 \cdot 10^{-3}$ secs. and the total conductivity to be equal to $5 \cdot 10^{-12}$, we get $I \cong 8 \cdot 10^{11}$ pairs of ions per sq cm sunbeam column per sec, which is about $\frac{8 \cdot 10^{17}}{2 \cdot 10^{12}} = 4 \cdot 10^{5}$ times too great.

Even if we assume that free electrons are only lost by recombination, which cannot be the case, the necessary electron density n_{e0} is $3.3 \cdot 10^{9}$ electrons per c c, and the corresponding value of I is $1.1 \cdot 10^{10}$ pairs of ions per c c per sec. The total number of ionizations per sq cm of sunbeam column per sec is then $1.1 \cdot 10^{16}$, or more than $5 \cdot 10^{3}$ times too great.

It thus appears that the energy of the ultraviolet solar radiation is far too small to maintain an ionization which may cause such a high conductivity at such a low altitude, even if we assume the electrons which are liberated by the solar ionization to remain free until they are lost by recombination with positive ions, which is altogether too favourable an assumption.

In our opinion, therefore, all hypotheses involving highly conductive »layers« at altitudes below 70 km are very improbable.

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[1] See Chapter IX., p. 148.

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(f) The Direction of the Horizontal Component of the Resultant Magnetic Field in the Waves.

In direction-finding by means of a small vertical loop rotating around a vertical axis a determination is made of the direction of the horizontal component H_0 of the resultant magnetic field, and the direction of propagation D of the waves may or may not be at right angles to this horizontal component. It is obvious that D cannot be determined from a measurement of H_0 unless the angle between D and H_0 is known. In practice it is often tacitly assumed that D and H_0 are at right angles to each other, although this only applies to the case where either the true direction of propagation or the resultant magnetic field or both are horizontal. In all other cases D will not be at right angles to H_0 and there will, according to circumstances, be a greater or smaller »error« in direction-finding by means of the simple loop method.

It should be noted, however, that: (1) A tilted wave front does not necessarily cause directional errors; in fact the wave front may be inclined at any angle provided the direction of its magnetic field remains horizontal. (2) In a vertical wave front the magnetic field may be directed at any angle without giving any directional error. (3) In the case of down-coming waves it is the direction of the resultant horizontal component of the magnetic field which is determined by the simple loop method and this resultant depends upon the magnetic fields in both the incident and the reflected wave and also upon the phase angle between the two. The direction in this case therefore depends on the height of the loop or coil above the ground[1].

We shall not enter here into any detailed discussion of the direction-finding problem but only refer to the publications of *R. L. Smith-Rose* and *R. H. Barfield*[2] and *R. Mesny*[3]. The first mentioned authors have recently shown (in paper (b)) that the direction of propagation of the waves generally very nearly follows the great circles through the transmitting station and the receiver and that most of the small deviations from this rule found in practice are due to local irregularities at the receiver. The difference between this true direction of wave-rays and the direction found by a simple coil direction-finder is due to the fact that down-coming waves have a component of the magnetic field in the plane of incidence. We shall consider this question briefly for the case of a simple coil direction-finder directly above a perfectly conducting ground. We assume two incoming waves, one wave S in Fig. 12 moving along the surface with a vertical wave front and with the magnetic field H at right angle to the plane of incidence, and another wave S_i coming down at an earth angle ψ and with the magnetic field H_i in the wave front and turned an angle α from the horizontal. The horizontal components of H_i are $H_i \cos\alpha$ perpendicular to and $H_i \sin\alpha \cdot \sin\psi$ in the plane of incidence. These horizontal components

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[1] For particulars about reflection of radio waves we may refer to Chapt. VIII, sect. 2 and to *L. Bouthillon:* Journ. de l'Ecole Polyt. 2es, 25e Cahier. p. 151—190. 1926.

[2] *R. L. Smith-Rose* and *R. H. Barfield:* (a) A Discussion of the Practical Systems of Direction-finding by Reception. (Radio Research Board, Special Report No. 1. London 1923).

(b): Journ. Inst. El. Eng. Vol. 64, p. 831—837. 1926.

R. L. Smith-Rose: Variations of Apparent Bearings of Radio Transmitting Stations. (Radio Research Board, Special Reports Nos. 2—4. 1924—26).

[3] *R. Mesny:* Usage des cadres et radiogoniométrie. (Paris 1925).

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will be doubled at the reflection, and the direction H_0 of the resultant horizontal magnetic field will be determined by

$$\mathrm{tg}\,\varphi = \frac{2\mathrm{H_i}\sin\alpha\cdot\sin\psi}{\mathrm{H} + 2\mathrm{H_i}\cos\alpha}. \tag{32}$$

φ is also the angle which the apparent direction D' makes with the true direction of propagation D of the wave[1] [2].

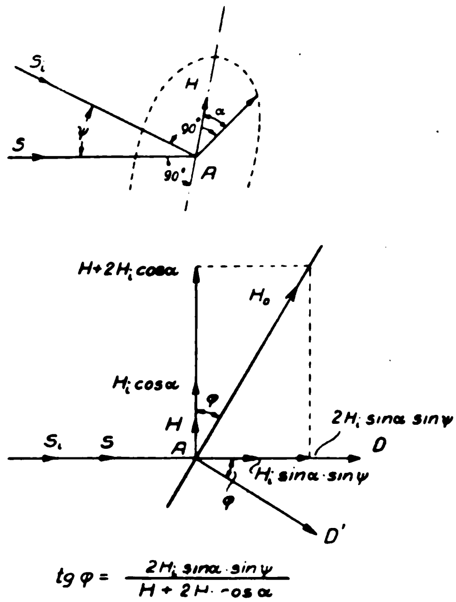

Fig. XI. 12. S a wave with vertical front moving along the surface, H being the horizontal component of the magnetic field. S_i a down-coming wave in which the magnetic field H_i forms an angle α with the horizontal line in the wave front. The horizontal components $H_i\cos\alpha$ and $H_i\sin\alpha\cdot\sin\psi$ are doubled by reflection. D the true travelling direction of the wave, D' the apparent direction as determined by the simple loop method.

From formula (32) it appears that $\varphi = 0$ for $\alpha = 0$ and that for $\frac{\mathrm{H}}{2\mathrm{H_i}} \to 0$ the formula reduces to

$$\mathrm{tg}\,\varphi = \sin\psi\cdot\mathrm{tg}\,\alpha. \tag{33}$$

For short waves and not very small distances the earth-bound wave will be practically nil and the directional error φ will have the value given by formula (33), i. e. $\mathrm{tg}\,\varphi$ will be proportional to $\mathrm{tg}\,\alpha$.

In Chapt. VII formula (57b) we found that for very short waves the plane of polarization was turned through an angle of 2π in traversing a distance $l_{2\pi}$ determined by

$$\mathrm{l}_{2\pi} = \frac{\mathrm{mc}}{\mathrm{N}\cdot\mathrm{e}^2}\cdot\frac{\omega^2}{\mathrm{h}}\cdot 10^{-5}. \tag{34}$$

With $N = 5\cdot 10^5$ electrons per c c this formula gives

$$\mathrm{l}_{2\pi} = 2.4\cdot 10^{-9}\cdot\frac{\omega^2}{\mathrm{h}}. \quad (\mathrm{km}) \tag{35}$$

If we assume the horizontal intensity of the magnetic field of the earth to be $H = 0.2$ gauss then the value of h is about $3.6\cdot 10^6$ and formula (35) becomes

$$\mathrm{l}_{2\pi} \simeq 7\cdot 10^{-16}\cdot\omega^2. \quad (\mathrm{km}) \tag{36}$$

For $\omega = 10^8$ we get $\mathrm{l}_{2\pi} = 7$ km. The value of α at the receiver will therefore be quite arbitrary. For long distance transmission even a variation of about 1 per cent in the value of the horizontal intensity of the magnetic field of the earth will cause several revolutions of the plane of polarization. Since

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[1] In the case of completely absorbing ground we get instead of (32) the following formula:

$$\mathrm{tg}\,\varphi = \frac{\mathrm{H_i}\cdot\sin\alpha\cdot\sin\psi}{\mathrm{H} + \mathrm{H_i}\cos\alpha} \tag{32a}$$

while formula (33) is unaltered.

[2] If the down-coming ray is either circularly or elliptically polarized a simple coil direction-finder gives either a flat minimum or no minimum at all.

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variations of this order of magnitude continually occur the short waves will generally show no direction effect at all, since the zero position of the coil changes so rapidly that no zero readings can be taken. This consequence is in agreement with the experimental evidence[1].

(g) Continuous Path Shifting, Echo effects, and Doppler effects.

Let *bac* in Fig. 13 I be the normal (n, h)-curve for day time and for the wave length considered, and let (Fig. 13 II) AA_0 be the normal (main) ray path from the transmitter A to the receiver at the distance R. (For the sake of sim-

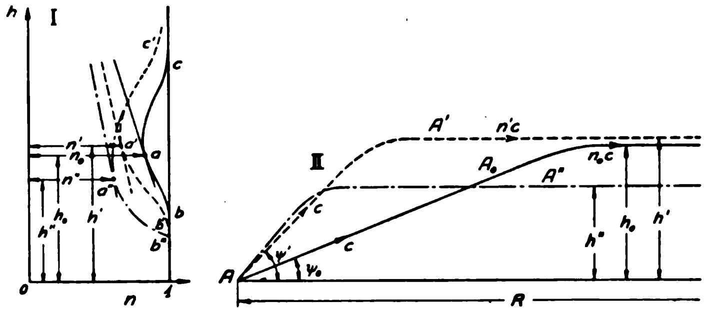

Fig. XI. 13. Diagrammatic Representation of Continuous Path Shifting.

plictiy we assume the earth to be flat). This ray AA_0 corresponds to the point a on the n-curve in which

$$-\frac{dn}{dh} = \frac{n}{\varrho} \tag{37}$$

where ϱ is the radius of curvature at the top of the path which is at the altitude h_0.

The earth angle ψ_0 of AA_0 is determined by

$$\cos\psi_0 = n_0, \tag{38}$$

n_0 being the refractive index at the apex of the ray.

If the ionization is increased in such a way that the resultant n-curve is represented by the dotted line $b'a'c'$, then the new normal ray-path AA' will have an earth angle ψ' determined by

$$\cos\psi' = n' \tag{39}$$

and will correspond to that point a' of the n'-curve in which

$$-\frac{dn'}{dh} = \frac{n'}{\varrho}. \tag{40}$$

The difference in path length is approximately equal to

$$\varDelta R \simeq 2h'\frac{1-n'}{\sqrt{1-n'^2}} - 2h_0\frac{1-n_0}{\sqrt{1-n_0^2}}. \tag{41}$$

[1] See f. inst.: *R. Mesny:* l. c., p. 32.

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The group-velocities at the tops of the two paths are approximately equal to $n_0 c$ and $n' c$ respectively (see Chapt. X, formulas (68) to (71)) and it is easily shown that the difference $\Delta T'$ between the time it takes for a signal to travel from A to the receiver along AA' and AA_0 respectively is approximately determined by:

$$\Delta T' \cong \frac{R}{cn'} - \frac{R}{cn_0} = R \cdot \frac{n_0 - n'}{cn_0 n'}. \tag{42}$$

$c \cdot \Delta T'$ is the difference in »equivalent lengths« of the two ray paths.

If for instance we take $R = 1200$ km, $n_0 = 0.90$ and $n' = 0.80$ we get $c \cdot \Delta T' =$ 167 km.

The extra-ionization may be even stronger and may penetrate down to altitudes considerably below *ab*. Such a case is shown by the dot and dash line $b''a''$. The ray AA'' corresponds to the a'' on this n-curve. The formula for determining $c \cdot \Delta T''$ will be similar to (42).

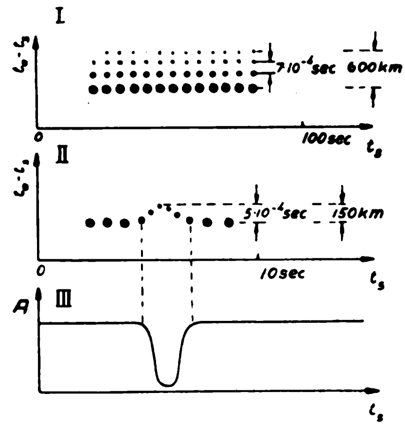

Fig. XI. 14. Diagrammatic Representation of Echo-effects and Continuous Path Shifting. t_s represents the time of sending the dots, t_a the time of arrival, $t_a - t_s$ consequently the time of transmission. The size of the dots indicates the strength of the received signals. Part I shows Echo-effects, part II a continuous changing of path and part III the strength A of the signals in the last case. The data in the figure refer to the experience of the Telefunken Company with regard to short-wave picture transmission between Berlin and Rome.

It follows from $n'' < n' < n_0$ that the electron densities are greater at the apex of the rays AA' and AA'' than at that of the ray AA_0, and the rays AA' and AA'' will therefore be attenuated much more than AA_0, or in other words, there will be a very strong fading during the deviation from the normal ray path. The question of attenuation will be discussed further under heading (i).

Such effects are observed for instance by the Telefunken company[1] and must be expected whenever the normal ionization is increased by a radiation which penetrates down to and below the lower level of the normal ionization. This penetrating radiation may be corpuscular rays from the sun or from elsewhere (compare *C. Störmer*'s measurements of the lower levels of auroras polaris in Fig. 8 of Chapt. V). At lower altitudes the electrons set free by this ionization will rapidly be lost by recombination and by being caught by neutral oxygen and water molecules. If the extraneous ionization is active only during a very short time the duration of the whole process may be only a few seconds. All these consequences are in agreement with the description of the phenomenon given by *Rukop*, compare Fig. 14.

If our view concerning the transmission problem is correct it is evident that there will be a possibility of a series of echos, their number, retardation and intensity depending on the circumstances. In Fig. 15. II. AA_0B is the normal ray path between A and B. Assuming the ground at the places A_2 and A_3 to be regular, plane and horizontal, there will also be a ray $AA_1A_2A_1A_3A_1B$ carrying

[1] *H. Rukop:* E. N. T. Bd. 3, p. 316—318. 1926.

energy from A to B. It is easily shown, however, that the time of transmission for a signal from A to B will be very nearly the same for the last mentioned path and for the normal path AA_0B. The same reasoning will apply to all possible paths lying below the normal path.

If, however, there is a strong extra ionization outside the normal ionization as shown by the corresponding n-curve *cab* in part I of Fig. 15, and if this extra ionization corresponds to the n-curve shown by the dotted line, then there will be two more normal rays connecting A and B, namely $AA'B$ and $AA''B$, corresponding to the points a' and a'' on the new n-curve. The equivalent length of these paths may be some hundreds of kilometres greater than the equivalent length of AA_0B. Also a fourth ray $AA'''A_2A'''B$ is shown which returns once to the surface at the point A_2, and we have in this case assumed

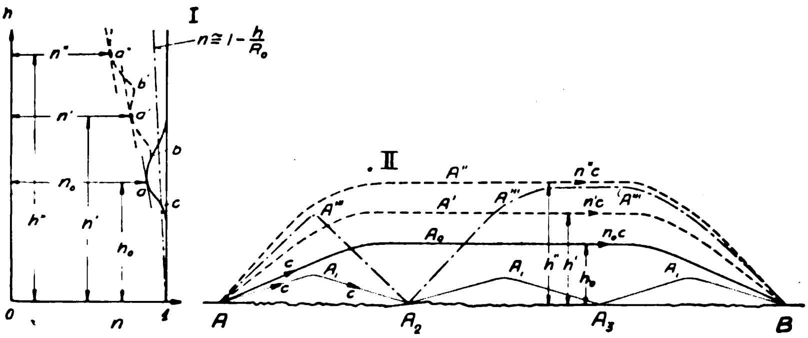

Fig. XI. 15. Echo-effects due to an Extra Ionization at High Altitudes.

that the plane surface at A_2 forms a small angle with the horizon. The equivalent length of this path will also be considerably greater than that of the path AA_0B.

These echos will be rather stationary for some time, since at the high altitudes assumed for the extraneous ionization the effect of recombination and the capture of the free electrons by neutral molecules is comparatively small.

Echo-effects depending upon an ionization outside the normal ionization will only be possible for wave lengths smaller than about 70 m, since for longer waves the normal n-curve even at night at the altitude of maximum ionization will approach zero so closely that these waves cannot reach such high altitudes. Compare Fig. 16, p. 216). This extra ionization may also be due to corpuscular rays. Echos of this kind are in many respects similar to those described by *Rukop*.

We have, under heading (e), mentioned some instances of echo effect due to the fact that one ray is following the shorter great circle path, the other ray the longer. When the transmitting and the receiving stations are not more than a few thousand kilometers apart the time-lag of this echo will generally be about $\frac{1}{7}$ sec. But this is only the normal value of the time-lag. Thus strong, local ionizations (due for instance to auroral rays) may cause great deviations from the normal value. For instance a strong, local ionization between the two stations will cause a great decrease in the minimum value of n and the earth angle of the long-step direct ray will accordingly be great. Owing to the strong ionization the group-velocity along the top part of the ray will be small, approximately equal to nc (see Chapt. X, formulas (55) and (68)-(71). The time-lag will therefore in this case be considerably less than $\frac{1}{7}$ sec.

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When conditions are favourable we may, on account of the influence of the earth's magnetic field on the group-velocity, expect even further complications in the echo phenomenon, the four group-velocities u_0, u_{I}, u_{II} and $u_{\perp}$ having different values and thus causing different timelags.

A continuous shifting of the path may give rise to Doppler effects which will make the frequency of the wave at the receiver different from that at the transmitter. The Doppler effect depends, however, on the phase-velocity and not on the group-velocity. For a path *AB* of length *L* and with the same refractive index *n* along the whole path, the number of waves between *A* and *B* is

$$N = \frac{f}{c} \cdot nL, \tag{43}$$

f being the frequency at the sending end of the path.

If the total length of the path is increasing in the ratio $\frac{dL}{dt}$ and the refractive index in the ratio $\frac{dn}{dt}$ then the frequency at *B* will exceed that at *A* by

$$-\frac{dN}{dt} = -\frac{f}{c}\left(L\frac{dn}{dt} + n\frac{dL}{dt}\right). \tag{44}$$

If we take $f = 1.5 \cdot 10^7$ (corresponding to a 20 m wave), $L = 10^4$ km, and $\frac{dL}{dt} = 0$, we get

$$\frac{dN}{dt} = 5 \cdot 10^5 \cdot \frac{dn}{dt}. \tag{45}$$

To $\frac{dN}{dt} = 10^3$ thus corresponds $\frac{dn}{dt} = 2 \cdot 10^{-3}$, or *n* must be increased by 0.01 in 5 seconds. Such alterations in the value of *n* may be due to variations either in the intensity of ionization or in the magnetic field of the earth. (See Chapt. X formulas (57), (58a), (63) and (65)). The first mentioned variation may be caused by corpuscular rays, by variations in the radiation from the sun, and possibly also by variations in the radiation from the stars. Such changes in frequency have been observed in many cases of long distance short-wave transmission, see for instance *H. Rukop* (l. c.).

If a signal wave (or an atmospheric disturbance) arrives at the receiver over two (or more) different ray paths along which the changes in frequency, $\frac{dN'}{dt}$ and $\frac{dN''}{dt}$, are different then the two rays will interfere with each other and give a note of a frequency $\frac{dN'}{dt} - \frac{dN''}{dt}$. This note will very often fall within the range of audibility. Effects of this kind are rather common, also with longer waves.

(h) Fading. Short Wave Scattering in the Atmosphere.

Fading may be due to variation in the attenuation caused by extra ionization due to variations in the ultra-violet radiation from the sun, or to corpuscular rays or the like. Fading may also be due to variations in the orientation of the plane of polarization[1] caused for instance by variations in the magnetic

[1] Or to variations in the state of polarization. The down-coming rays will in most cases be elliptically polarized.

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field of the earth. In cases where two or more rays arrive at the receiver[1], fading may also be due to interference between the various waves and this interference may be altered considerably by very small alterations in the electrical state of the atmosphere, or in the magnetic field of the earth.

We have up to the present assumed the electrical state of the atmosphere to be the same at all points at the same level. It is very probable, however, that the surfaces for which n and σ are constant may not be perfectly spherical but may have more or less pronounced ripples or waves; compare for instance the luminous night clouds mentioned below. The movements taking place in such a rippled or waved surface will also cause alterations in the conditions of the wave transmission.

According to the view of short-wave transmissions which we have described, it is to be expected that the field strength within the skip distance will be small at sufficiently great distances from the transmitter. It must be remembered, however, that the atmosphere is not absolutely continuous. The existence of luminous night clouds[2] shows that surfaces of discontinuities may exist at altitudes of about 80 km. Some discontinuities in the n-curve may therefore also be expected. In the lower atmosphere surfaces of discontinuity are of very common occurence; see Chapt. X sect. 1. Such surfaces showing discontinuities in the values of n will cause reflections which according to formulas (77) to (81) and Fig. 20 in Chapt. VIII are determined by

$$\left(\frac{a_r}{a_i}\right)_I = \frac{2\cos^2\varphi - 1}{2\cos^2\varphi}\cdot\Delta n \quad \text{and} \quad \left(\frac{a_r}{a_i}\right)_{II} = \frac{1}{2\cos^2\varphi}\cdot\Delta n, \tag{46}$$

where $\Delta n = n_0 - n$ and where it is assumed that $\cos\varphi \gg \Delta n$.

If we take Δn equal to $2.5\cdot 10^{-4}$ (see Chapt. X, sect. 1) then the ratio $\left|\frac{a_r}{a_i}\right|$ will in many cases exceed $2\cdot 10^{-3}$. Since several surfaces of discontinuity may be effective at the same time, and in consideration of the evidence of Chapt. VIII, sect. 4, especially Table 6 and Fig. VIII, 20, there is no difficulty in explaining the fact that rather strong short wave signals may often be heard within the skip distance and at some distance from the transmitter.

Furthermore there may also be surfaces at which the value of n varies continuously while there is a discontinuity in the values of $\left(\frac{dn}{dh}\right)$ — or more generally in $\left(\frac{dn}{dx}\right)$. This kind of discontinuity may also cause reflection which in the case of normal incidence, according to formula (76) Chapt. VIII, is given by:

$$\left|\frac{a_r}{a_i}\right| = \frac{\lambda}{8\pi n}\cdot\left[\left(\frac{dn}{dx}\right)_1 - \left(\frac{dn}{dx}\right)_2\right]. \tag{47}$$

If we take

$$\lambda = 25\ \text{m},\quad n = 1 \quad \text{and} \quad \left(\frac{dn}{dx}\right)_1 - \left(\frac{dn}{dx}\right)_2 = 2\cdot 10^{-4}\,\text{m}^{-1} \quad \text{we have} \quad \left|\frac{a_r}{a_i}\right| \cong 2\cdot 10^{-4}.$$

---

[1] In the case of down-coming short waves there will always be a beam of rays with (slightly) different earth angles and with different states of polarization (see sect. 3 p. 229).

[2] See for instance *A. Wegener:* Thermodynamik der Atmosphäre, p. 20 (Leipzig 1911).

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Both kinds of discontinuity may be met at any altitude, but we believe that the first mentioned kind is more common in the lower atmosphere and is one of the main causes of this short wave scattering[1]. An important source of down-coming rays is the reflection taking place within the lower part of the ionized region when the angle of incidence is but little smaller than the critical angle, see Chapt VIII, sect. 4.

Another possible cause of the relatively strong signals found at times inside the skip distance may also be mentioned, namely the scattering taking place when down-coming waves are reflected from broken ground, mountains or the like. In that case some of the »rays« may be reflected backwards and come down again within the skip zone[2].

(i) General Remarks. Attenuation. Variations of the Short-Wave Limits.

From the presented theory of short-wave transmission and from the n- and $\left(\frac{1}{\gamma_0}\right)$-curves shown in Figs. IX. 13—18 and 20 (in the appendix) it is possible to draw several further conclusions with regard to short-wave propagation. We shall, however, here consider only a few of the most important of these conclusions.

From sect. 2(d) of this chapter it may be seen that the greatest part of an effective long-distance ray is at an altitude slightly below the altitude of maximum electron density, and the attenuation constant γ_0 may in this case, where $4\pi c^2\sigma \ll \omega\varepsilon$ and $\varepsilon > 0$, according to formula (8a) in Chapt. VIII be determined by

$$\gamma_0 = N \cdot \frac{2\pi e^2}{mcn_0} \cdot \frac{v}{v^2 + \omega^2} \cdot 10^5 \simeq N \cdot \frac{2\pi e^2}{mcn_0} \cdot \frac{v}{\omega^2} \cdot 10^5, \quad [\text{km}^{-1}] \tag{48}$$

where N is the number of free electrons per c c, the ions being of no importance in this case.

The corresponding value of the refractive index n_0 is

$$n_0 = \sqrt{\varepsilon} = \sqrt{1 - N \cdot 4\pi \frac{e^2}{m} \cdot \frac{1}{v^2 + \omega^2}} \simeq \sqrt{1 - N \cdot 4\pi \frac{e^2}{m} \cdot \frac{1}{\omega^2}}, \tag{49}$$

and the earth angle ψ is as usual determined by

$$\cos \psi = \left(1 + \frac{h}{R_0}\right) \cdot n_0 \simeq \left(1 + \frac{h}{R_0}\right) \cdot \sqrt{1 - N \cdot 4\pi \frac{e^2}{m} \cdot \frac{1}{\omega^2}}. \tag{50}$$

It appears from (48) that if the electron density N is either constant or so small, or the frequency so great, that the value of n_0 does not differ considerably from unity, then the attenuation constant is proportional to the electron density and to the square of the wave length.

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[1] *H. Fassbender, K. Krüger* und *H. Plendl:* Die Naturwissenschaften. Bd. 15, p. 357. 1927.
E. V. Appleton: »Electrician«, Vol. 98, p. 256—257. 1927.
T. L. Eckersley: l. c.

[2] It should be remembered that the published results of most of the measurements of field strengths are the average of a number of measurements. This process of averaging will tend to weep out any discontinuity which may exist in the individual curves representing the field strength as a function of the distance.

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For very short waves we therefore have

$$\gamma_0 \cong a \cdot N \cdot \lambda^2. \tag{51}$$

This dependency of the attenuation upon the wave length is in agreement with experience in short wave transmission[1]. The proportionality between γ_0 and N also agrees well with the fact that the attenuation of the short waves is greater at times of sun-spot maximum than at times of sun-spot minimum[2], and is greater on days with magnetic »storms« than on magnetically quiet days. N is of course assumed to be greater in the first mentioned cases.

Formula (51) also agrees with the fact that the attenuation in long-distance short-wave transmission is less by night than by day, the value of N in the ray path being greatest by day.

Variations in the short-wave limits λ_{RDmin} and λ_{RNmin} may be due to auroral rays (corpuscular rays), to variation in the electron (and ion) density caused by variations in the ultra-violet radiation from the sun, to variations in the earth's magnetic field, etc.

We shall here mainly discuss the influence of the variations of the solar radiation. According to *Chapman* the ionizing power of the sun is from 2 to 2.5 times greater at times of sun-spot maximum than at times of minimum. We shall therefore determine the influence of increases of 100 and 150 per cent in the ionizing power of the sun upon the values of the short-wave limits.

At the altitude of maximum electron density by day the value of N, according to formula (38) in Chapt. V, is proportional to I, the number of ionizations per c c per sec., namely

$$N = I\tau, \tag{52}$$

where τ is the mean free time of an electron.

We shall therefore consider the two electron densities $N = N_0$ and $N' = 2N_0$. The corresponding short-wave limits λ_{RDmin} and λ'_{RDmin} are determined by the condition that the value of n shall be the same in both cases, i. e.

$$n_0 \cong \sqrt{1 - N_0 K \cdot \lambda^2_{RDmin}} = \sqrt{1 - 2N_0 \cdot K \cdot \lambda'^2_{RDmin}}, \tag{53}$$

or

$$\lambda'_{RDmin} = \frac{\lambda_{RDmin}}{\sqrt{2}} \cong 0.7\, \lambda_{RDmin}. \tag{54}$$

The corresponding night densities of electrons will according to formula (45) in Chapt. V have a ratio of about 1.5, and the ratio of the short-wave limits will be

$$\lambda'_{RNmin} = \frac{\lambda_{RNmin}}{\sqrt{1.5}} \cong 0.82\, \lambda_{RNmin}. \quad (55$$

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[1] *T. L. Eckersley:* l. c., Figs. 39 and 40.

[2] According to *S. Chapman* (Phil. Trans. (A). Vol. 218, p. 48, 1919) the ionizing power of the sun is about 2 to 2.5 times greater at times of sun-spot maximum than at times of minimum.

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Increases of 100 and 150 per cent in the ionizing power of the sun will thus reduce the short-wave limits from

$$\lambda_{RDmin} = 8.5 \text{ m} \text{ and } \lambda_{RNmin} = 18.9 \text{ m}, \quad \text{to}$$

$$\lambda_{RDmin} = 6.0 \text{ m} \text{ and } \lambda_{RNmin} = 15.5 \text{ m},$$

for an increase of 100 per cent, and to

$$\lambda_{RDmin} = 5.4 \text{ m} \text{ and } \lambda_{RNmin} = 14.5 \text{ m}^{1},$$

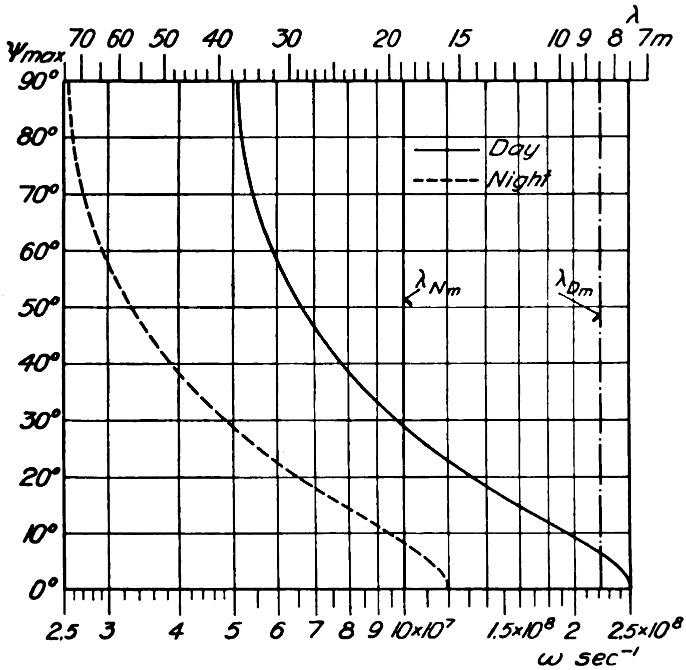

Fig. XI. 16. Earth Angle ψ_{max} of effective Long-Distance Ray as a Function of Wave Length (or Frequency) for Summer Day and Summer Night.

for an increase of 150 per cent. We have here assumed that the short-wave limits $\lambda_{RDmin} = 8.5$ m and $\lambda_{RNmin} = 18.9$ m correspond to times of sun-spot minimum.

[1] If earth angles down to zero are considered — and such small earth angles may in some cases be effective in one-step transmissions — the last mentioned limits are further reduced to $\lambda_{RDmin} = 4.7$ m and $\lambda_{RNmin} = 12.1$ m. If our values of the normal short-wave limits are correct, these last mentioned values are the smallest possible ones which may be obtained with solar ionization alone.

Auroral (corpuscular) radiations may cause further transitory decreases in the values of the short-wave limits.

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Other important questions are: How does the earth angle of the effective long-distance rays depend upon the wave length? What are the upper day and night limits of the length of the waves which may give effective long-distance transmissions?

These questions may very easily be answered, for instance for summer time, by means of Fig. IX, 14 (in the appendix), and we have in Fig. 16 shown the earth angle, ψ_{max}, of the effective long-distance rays as a function of the wave length (or the frequency) for day and night in summer, *i.e.* the greatest values of ψ possible with the wave length considered.

It appears from this figure that the earth angles by day for very short waves, from 10 to 15 m, are from 10° to 20°, for a 20 m wave about 30°, for a 30 m wave about 53°, while waves for which $\lambda >$ about 37 m cannot be regularly transmitted in the long-step manner here considered.

By night 20 to 30 m waves have earth angles between 10° and 20°, and $\lambda = 40$ m corresponds to $\psi_{max} \simeq 30°$, $\lambda = 60$ m to $\psi_{max} \simeq 55°$, while all waves for which $\lambda >$ about 71 m cannot be regularly transmitted in this manner.

The values of the earth angles agree very well with the experimental values found by *Eckersley*[1] and others.

We shall here also quote some results found by *Breit* and *Tuve*[2]: »Signals on 71.3 meters showed reflections ··· in great majority of cases. On 41.7 meters at $10^h 30^m$ A. M. a number of tests in August and September failed to show reflections.···· No reflections were observed on 20 meters or shorter wave-lengths«.

With the nearly normal incidence used in *Breit* and *Tuve*'s experiments the shortest wave which is regularly reflected by day is, according to Fig. 16, about 37.5 m. At times of sun-spot maximum this value may be reduced to

$$\lambda_0 = \frac{37.5}{\sqrt{2.5}} \simeq 23.8 \text{ m}.$$

According to Fig. 16 waves for which $\lambda > 71$ m will always be reflected. Waves just a little greater than 37.5 m will only be regularly reflected at noon in summer time, while waves down to 23.8 m may occasionally be reflected; these predictions are in good agreement with *Breit* and *Tuve*'s results.

Waves which are longer than the two upper limits, 37 m by day and 71 m by night, cannot be transmitted over long distances in the long-step manner. But for very long distances the effective upper limits will be smaller than the figures given, the determining factor in this case being the attenuation.

The energy of waves longer than 37 m and 71 m, respectively, cannot be lost by the waves escaping into free space, and such waves are effectively enclosed within the space between the earth's surface and the ionized region of the atmosphere. The electrical energy of these waves may be lost at the surface of the earth, within the lower part of the ionized region and within the atmosphere between the earth and the highly ionized layer. The last mentioned loss is due to the feeble ionization found in this part of the atmosphere and is without any appreciable influence for the short waves considered here.

[1] *T. L. Eckersley:* l. c., p. 607.

[2] *Breit* and *Tuve:* l. c., p. 556.

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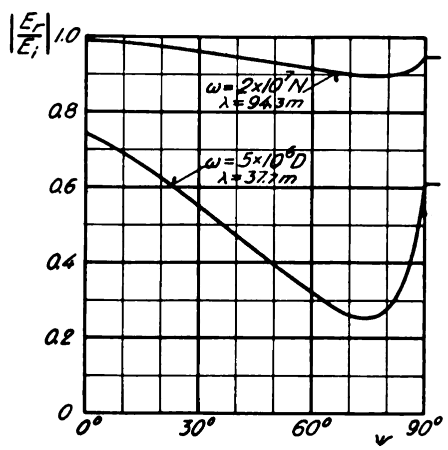

Fig. XI. 17. The Ratio $\left|\frac{E_r}{E_i}\right|$ of the Energy of the Emergent Ray to that of the Incident Ray as a function of the Earth Angle ψ for a 37.7 m Wave by Day and a 94.4 m Wave by Night. This ratio is calculated by means of the formulas in Chapt. X, sect 5 assuming Γ to have twice the value determined by formulas (83) to (86).

For waves for which λ by day > 37 m and λ by night > 71 m and for which $\lambda < 100$ m, the transmission must take place in the short-step manner already discussed. An inspection of Fig. IX. 14 (in the appendix) shows that for small earth angles — ψ less than 20^0 — the loss in the lower boundary of the highly ionized layer will be relatively small. As experience shows that such waves are not regularly effective over very long distances the reason must be that the losses caused by the earth, directly and indirectly[1], are too great.

In this connection it may be of some interest to see how the loss within the lower part of the ionized region depends upon the value of the earth angle and we have therefore in Fig. 17 shown the ratio of the energy of the emergent ray to that of the incident ray as a function of the earth angle for a 37.7 m wave by day and for a 94.4 m wave by night. It appears that this loss, as stated above, is quite small for small earth angles but that it increases rapidly with increasing earth angle.

We believe that we are justified in stating that all the conclusions drawn from the theory of radio wave propagation presented here are in fair agreement with the available experience in regard to short wave transmission.

3. *Propagation of Long Waves. ($\lambda > 400$ m).*

(a) General Considerations and Remarks.

With very short waves the ground waves (see Fig. 18) are attenuated so strongly that they do not play any rôle at all outside the immediate vicinity of the transmitting station (see Fig. 1), the effective long-distance radiation energy being carried by the free waves. With long waves the earthbound waves

[1] Under indirect losses we include, for instance, scattering from the surface of the earth in such directions that the energy of the waves is partially lost for the transmission under consideration.

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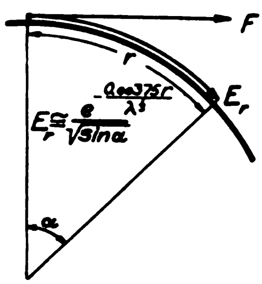

Fig. XI. 18. E_r is the Earthbound Wave or Ground Wave, F the Free Wave.

are of more importance. According to Chapt. II, formula (1), their field strength E_b in the distance b is approximately determined by

$$E_b \cong K \cdot \frac{e^{-\frac{0.00375}{\lambda^{1/3}} \cdot b}}{\sqrt{\sin\alpha}} \qquad \text{(b and } \lambda \text{ in km)}, \quad (56)$$

where K is a constant and α the angular distance (see Fig. 18).

Fig. 19 shows the value of $\frac{E_b}{E_{100}}$ for the ground waves according to formula (56) for various angular frequencies as a function of the distance b.

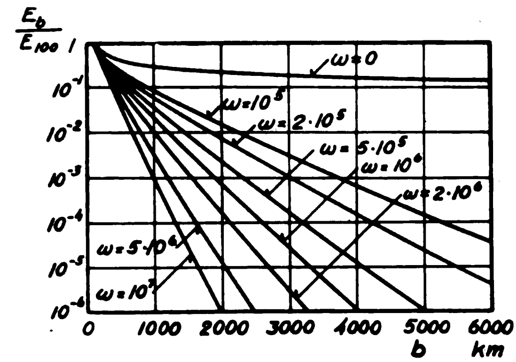

Fig. XI. 19. The values of $\frac{E_b}{E_{100}}$ for the Ground Waves according to formula (56) as functions of the distance b for various angular frequencies.

The variations of the mean value of the actual field intensity with varying wave length and distance is now known with some degree of certainty. This is due to the long series of observations made by the American Telephone and Telegraph Company, The Radio Corporation of America, The Marconi Company, the English Post Office, the French Government station at Meudon, the German Telegraph Administration and the Telefunken Company, the Federal Telegraph Company and many others, but especially to the incessant work of *L. W. Austin* (the Bureau of Standards). For daylight signals over salt water the mean value of the field intensity may, according to *Austin*[1], with fair accuracy be determined by

$$E_b = 120\pi \cdot \frac{H_s I_s}{\lambda b} \cdot \sqrt{\frac{\alpha}{\sin\alpha}} \cdot e^{-\frac{0.0014}{\lambda^{0.6}} \cdot b} \qquad \text{(volts, km, amp.)} \quad (57)$$

where I_s is the current in and H_s the effective height of the transmitting antenna. This formula is approximately accurate for all wave lengths between 300 m and 25 000 m and for distances greater than about 1000 km.

Fig. 20 shows the value of $\frac{E_b}{E_{100}}$ according to *Austin*'s formula (57), while Fig. 21 shows the ratio between the intensity of the ground waves according to formula (56) and of the actual waves according to *Austin*'s formula. It appears from this figure that for distances of more than 1000 km and for wave lengths less than 5000 m only a small part of the received energy is due to the direct, earthbound wave. For great distances and not too long waves practically the

[1] *L. W. Austin:* (a) Journ. Wash. Ac. Sc. Vol. 16, p. 228, 1926.
(b) Proc. Inst. Rad. Eng. Vol. 14, p. 377, 1926.

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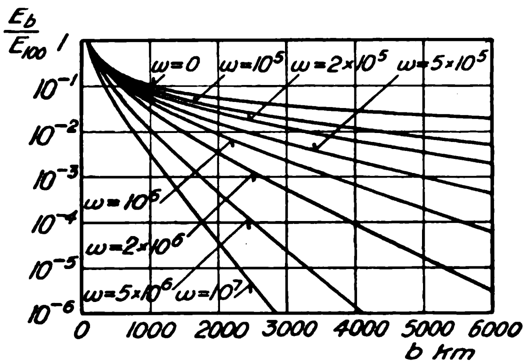

Fig. XI. 20. $\frac{E_b}{E_{100}}$ as a function of the distance b according to *Austin*'s formula (57).

whole transmitted energy is due to the free waves and these waves must necessarily be refracted or reflected down to the surface of the earth. The actual ray-paths may make one or more steps, compare sect. 2 (d).

From an inspection of the Figs. 11, 13, 15 and 17 in Chapt. VIII it appears that for long waves the reflection loss at the earth's surface will generally be small, and always smallest for long waves with the electric field at right angles to the plane of incidence. For long waves with the electric field in the plane of incidence there may be considerable losses for nearly grazing incidence.

At the upper boundary of the shell in which the long-wave transmission takes place, the value of the refractive index n is varying so rapidly with the altitude (see Figs. IX, 13—18 in the appendix) that there is a considerable change in the value of n within a difference in altitude of one wave length, at least for the very long waves. In these cases there will be some reflection. In nearly all cases a considerable part of the energy of the incident wave will again be refracted down to the earth's surface as shown in Chapt. X. Even for rays having large earth angles a great part of the incident energy in most cases will again be refracted downwards. There is, however, an exception for very long waves. From Figs. IX. 13—17 (in the appendix) and Fig. IX, 19 it appears that for a $\omega = 10^5$ wave the refracted rays will be almost completely attenuated for all earth angles greater than about 55°. For the $\omega = 10^6$ wave the corresponding angle is about 80° and for the $\omega = 2 \cdot 10^4$ wave about 45°. For smaller earth angles the refracted rays will be relatively strong[1].

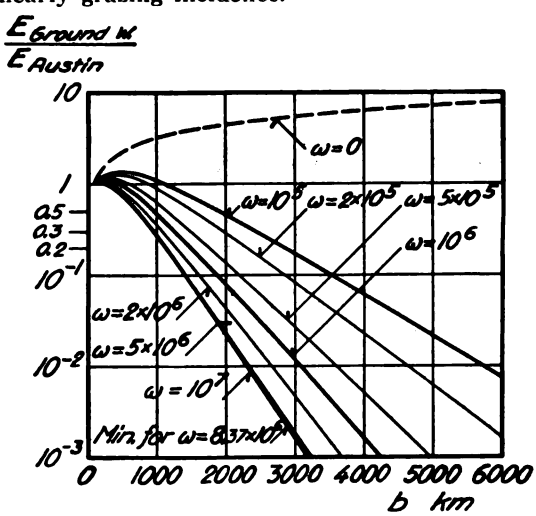

Fig. XI. 21. Ratio of $\frac{E_b}{E_{100}}$ for the ground wave according to formula (56) and for the actual wave according to *Austin*'s formula.

[1] The experimental results of *R. L. Smith-Rose* (Special Reports Nos. 3 and 4) indicate that for long waves ($\lambda = 12\,400$ m) there is no appreciable down-coming rays at distances from the transmitting station less than about 50 km while there is a strong down-coming ray for distances over 80 km (Report No. 3 p. 104 and No. 4 p. 13). This is in good agreement with our calculations.

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This sudden decrease in the intensity of the refracted ray for a small increase in the earth angle is due to the extra minimum in the value of the refractive index, which exists only in case of the long waves. Any ray having an earth angle great enough to pass this lower minimum will penetrate to such an altitude that the refractive index has a sufficiently small value. The corresponding difference in altitude will for the $\omega = 10^5$ wave be about 90 km, and since the value of the attenuation constant γ_0 for the greatest part of this distance is greater than $1\,\mathrm{km}^{-1}$, the refracted ray will be completely obliterated. There will therefore also be a sudden alteration in the down-coming energy when the earth angles pass over these critical values, since for smaller earth angles there are both the reflected and the refracted rays while for greater angles we have solely the reflected ray.

There is thus a possibility of an experimental determination of the value of these lower minima of the refractive index for the long waves; and valuable information on the state of ionization at these altitudes may thus be obtained.

(b) Long Wave Transmission over Short Distances, i. e. up to about 1000 km.

Up to about 1000 km for the very long waves (20 000 m) and to about 400 km for the shorter waves (300 to 600 m) the field intensity depends both on the ground wave and on the reflected waves, the field at any point being the resultant of the fields of all these waves.

The intensity of the ground wave may be determined by means of formula (56) or by Fig. 19. Besides the ground wave there may be a number of reflected and refracted waves and theoretically this number may be very great, but practically only very few of these refracted or reflected rays will be of any importance. We shall consider this case a little more closely.

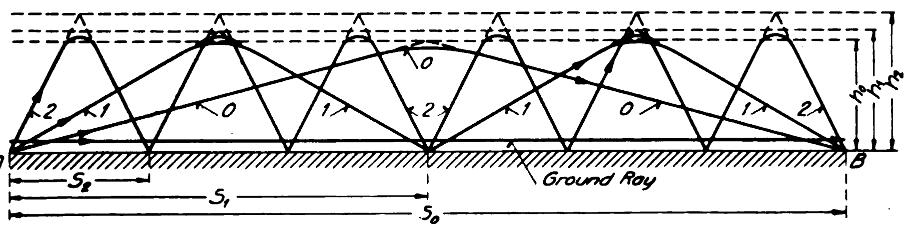

Fig. XI. 22. Diagrammatical representation of the Ground Ray and three »Stepping« Rays, 0, 1 and 2.

In Fig. 22 A is the transmitting station and B the receiver. A ground wave is going directly from A to B, while the ray 1 makes two, the ray 2 six »steps« before arriving at B. We assume that the ratio of the amplitude in the reflected to that in the incident ray is e^{-a_e} for reflection at the earth's surface while the ratio of the down-coming to that of the incident ray is equal to e^{-a_u} at the reflecting and refracting »layer«.

Between this »layer« and the surface of the earth there is a slightly conducting layer owing its conductivity mainly to the highly penetrating cosmic radiation mentioned in Chapt. V, sect. 5 (see also Fig. IX. 11 in the appendix). This conductivity below an altitude of about 80 km is of no importance with

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regard to the propagation of the short waves but is one of the main causes of the attenuation of very long waves[1].

A ray such as 0, 1 or 2 (see Fig. 22) will for each step of the ray have a length Δl of path within a layer of thickness Δh (km) determined by

$$\Delta l = \Delta h \cdot \sqrt{\left(\frac{S}{h}\right)^2 + 4}, \tag{58}$$

where S is the length of the step, and h the apparent height of the »reflective« layer.

Let the conductivity of the layer Δh be σ, the attenuation constant within this layer is then approximately determined by formula (8) in Chapt. VIII, or

$$\gamma_0 = 6\pi\sigma \cdot 10^{15}, \qquad [\mathrm{km}^{-1}] \tag{59}$$

since in this case $\varepsilon \cong 1$ and $\sigma \ll \frac{\omega\varepsilon}{4\pi c^2}$. The layer Δh will accordingly within one step reduce the amplitude of the ray in the ratio $e^{-\Delta a_i}$, where

$$\Delta a_i = \Delta l \cdot \gamma = 6\pi \cdot 10^{15} \cdot \sigma \Delta h \cdot \sqrt{\left(\frac{S}{h}\right)^2 + 4}. \tag{60}$$

The total attenuation of this kind within one step is accordingly

$$a_i = 6\pi \cdot 10^{15} \cdot \sqrt{\left(\frac{S}{h}\right)^2 + 4} \cdot \Sigma\sigma \cdot \Delta h = 6\pi \cdot 10^{10} \cdot \sqrt{\left(\frac{S}{h}\right)^2 + 4} \cdot G, \tag{61}$$

where G is the total conductivity of the intermediate layer [in cm·e. m. u].

The total effective attenuation constant is accordingly

$$\alpha = \alpha_e + \alpha_u + \alpha_i = \frac{a_e}{S} + \frac{a_u}{S} + 6\pi \cdot 10^{10} \cdot \sqrt{\frac{1}{h^2} + \frac{4}{S_2}} \cdot G. \qquad [\mathrm{km}^{-1}] \tag{62}$$

It is evident that for $S \to 0$ we have $\alpha_e \to \infty$, $\alpha_u \to \infty$ and $\alpha_i \to \infty$. Thus even if the long waves theoretically may take very short steps, the attenuation is very great for this manner of propagation, and the short steps can only be of any importance in short distance transmission.

For such short distances the variation of field intensity with the distance will generally not at any particular season be in accordance with *Austin*'s formula but will, owing to inference between the ground ray and the down-coming ray or rays (as the rays 0, 1 and 2 in Fig. 22), show systematical deviations from *Austin*'s values. Of the down-coming rays the one step ray 0 will generally give much stronger interference than the rays 1 or 2. Firstly because ray 0 is generally the strongest, secondly because its angle with the ground ray is the smallest.

This interference has been beautifully proved by *J. Hollingworth*[2], and Fig. 23[3] shows the results from one series of experiments for summer time. At other

[1] It is a mistake when *W. Kolhörster* (Die Naturwissenschaften, Bd. 15, p. 126, 1927) believes that the relatively small (10 to 15 per cent) variations in the highly penetrating radiation in the course of the stellar day may cause measurable variations in the propagation of *short* waves. It is to the very long waves we have to look for any pronounced effect of this kind.

[2] *J. Hollingworth:* Journ. Inst. El. Eng. Vol. 64, p. 579—595, 1926.

[3] Fig. 23 is a somewhat simplified reproduction of Fig. 5 in *Hollingworth*'s paper.

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seasons the intensity curve is different and the mean values agree fairly well with *Austin*'s formula.

At a certain distance, *d*, for each wave length the vertical electric force produced by the down-coming waves will be equal to that of the direct earthbound wave. For shorter distances the ground wave is the stronger while for greater distances the down-coming wave predominates[1].

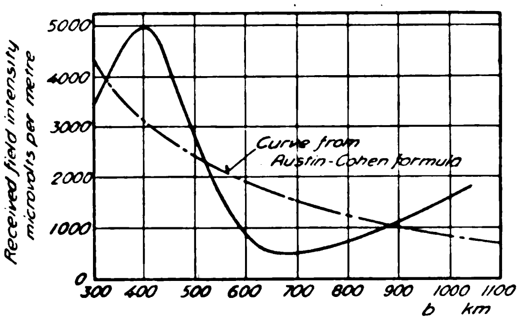

Fig. XI. 23. Average field intensity as a function of the distance for Summer Time and for $\lambda = 14\,350$ m.

c) Long Wave Transmission over Long Distances, i. e. over more than about 1000 km. Comparison with Austin's formula.

For long distance transmission the direct ground wave is of minor importance, and according to formula (62) the short step rays are also too strongly attenuated to be of any importance. Practically the whole transmission energy is therefore carried by the long step rays. In this case we may assume the radiation energy to be practically evenly distributed between the earth and the »reflecting layer«. We thus get a case which is similar to that treated in Chapt. III, p. 28—32.

In order to compare our theoretical values with *Austin*'s formula it must be remembered that we assume the radiation energy to be enclosed between the surface of the earth and the »reflecting layer«. Our long wave transmission formula may therefore be written

$$E = k \cdot \frac{1}{\sqrt{b}} \cdot \sqrt{\frac{2}{3h_0}} \cdot e^{-\alpha b}, \tag{63}$$

while *Austin*'s formula — for day transmission over salt water — has the form:

$$E_A = k \cdot \frac{1}{b} \cdot e^{-\alpha_A b}, \tag{64}$$

where k has the same value in the two formulas. b is the distance and h_0 the height of the »reflecting layer«.

If the two formulas (63) and (64) shall give the same values af E for all distances we must have

$$\alpha = \alpha_A + \alpha^0 = \alpha_A + \frac{1}{2b} \lg n \frac{2b}{3h_0} \quad \text{or} \quad \alpha^0 = \frac{1}{2b} \lg n \frac{2b}{3h_0}. \tag{65}$$

The values of α^0 for $h_0 = 100$ km[2] and $b = 1000$ and 5000 km are given in the

[1] *E. V. Appleton* and *J. A. Ratcliffe:* (Proc. Roy. Soc. (A). Vol. 115, p. 293, June 1927) state that to $\lambda = 400$ m corresponds a value of d between 160 and 250 km, while for $\lambda = 1600$ m d is between 600 and 1000 km.

[2] The value of h_0 depends on the wave length and to some extent on the earth angle of the ray. $h_0 = 100$ km is only a rough mean value.

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following table. This table also contains the values for $\omega = 10^5$ and $\omega = 10^6$ of α'_e for $\sigma = 10^{-11}$ (salt water) and of α''_e for $\sigma = 10^{-14}$ (ordinary soil). These values are taken from Chapt. III, Fig. 9. The values of α_i are calculated by the formula (see formula (62)):

$$\alpha_i = \frac{6\pi 10^{10}}{h_0} \cdot G. \qquad [\mathrm{km}^{-1}] \qquad (66)$$

The values af a_u (see formula (62)) have been calculated in accordance with Chapt. X, sect. 5, and α_u has been determined by

$$e^{-2\alpha_u} = \text{eff. mean value of } \cos^3\psi \cdot e^{-\frac{2a_u}{S}}. \qquad (67)$$

All these values together with *Austin*'s attenuation constants have been collected in Table 10. It appears from this table that the values of the calculated attenuation constants — α' for transmission over salt water and α'' for transmission over land — agree fairly well with the value of $(\alpha_A + \alpha^0)$ calculated according to formula (65).

Too much weight is not to be given to this agreement partly because the calculation of α_u may be seriously wrong and partly because very little is known about the total conductivity G (see formula (61)) of the intermediate layer and the value of α_i therefore also is very uncertain. But it appears at least from the table that as far as the present knowledge goes there may be agreement between our attenuation values and those according to *Austin*. And it will hardly be possible to go further at present.

Furthermore we may from Table 10 and from the theory underlying it draw a number of conclusions which are in qualitative agreement with experience and which, at least in our opinion, give some weight to our point of view. We shall mention some of these conclusions:

For very long waves ($\lambda > 20\,000$ m) the ground attenuation α_e is so small even for transmission over land that there will be very little difference between land and salt water attenuation, while this difference is quite pronounced for shorter waves ($\lambda < 6000$ m). The difference between very long waves and short waves will for hilly or mountainous land be even greater than according to Table 10 and to Chapt. III, Fig. 9, because scattering will cause greater losses with the shorter waves. This agrees with experience[1].

The attenuation constant α_u due to the loss in the upper atmosphere is greater by day than by night but the difference between day and night values decreases with increasing wave length. For $\omega = 10^5$ the ratio E_{night}/E_{day}, calculated in the above mentioned manner, is about 1.3 for a distance of 1000 km and about 1.7 for 5000 km. This result also agrees fairly well with experience. For these very long waves this difference is due mainly to the number of electrons, which are set free by solar ionization, and exist in that lowest part of the highly ionized region through which the down-coming long waves have passed. This gives rather sharp limits within which the altitude of the apex of the down-coming long wave rays must necessarily fall. If this altitude were 10 km less than that assumed in Chapt. IX, this number of free electrons in day time at the apex of the ray paths would be so small, that their disappearance at night would make no observable alteration. And if this altitude were 10 km higher than assumed in Chapt. IX the difference between day and night attenuation for these very long waves would be too great.

[1] See for instance *W. Austin:* l. c. (a) p. 229.

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Table 10. Attenuation Constants. Summer Day. ($h_0 = 100$ km).

ω	b	α'_e $\sigma = 10^{-11}$	α''_e $\sigma = 10^{-14}$	α_i *	α_u **	$\alpha' = \alpha'_e + \alpha_i + \alpha_u$	$\alpha'' = \alpha''_e + \alpha_i + \alpha_u$	α_A	α^0	$\alpha_A + \alpha^0$	$\frac{\alpha' - (\alpha_A + \alpha^0)}{\alpha_A + \alpha^0}$
	km	km^{-1}	km^{-1}	km^{-1}	km^{-1}	km^{-1}	km^{-1}	km^{-1}	km^{-1}	km^{-1}	
10^5	1000	$0.03 \cdot 10^{-4}$	$1 \cdot 10^{-4}$	$3 \cdot 10^{-4}$	$12 \cdot 10^{-4}$	$15 \cdot 10^{-4}$	$16 \cdot 10^{-4}$	$2.4 \cdot 10^{-4}$	$9.5 \cdot 10^{-4}$	$11.9 \cdot 10^{-4}$	+ 0.33
	5000	0.03 · »	1 · »	3 · »	4 · »	7 · »	8 · »	2.4 · »	3.5 · »	5.9 · »	+ 0.19
10^6	1000	0.1 · »	3 · »	0.4 · »	21 · »	21.5 · »	24.4 · »	9.6 · »	9.5 · »	19.1 · »	+ 0.13
	5000	0.1 · »	3 · »	0.4 · »	9 · »	9.5 · »	12.4 · »	9.6 · »	3.5 · »	13.1 · »	— 0.27

* Below 84 km the value of σ is taken as half of that shown in Fig. IX, 11 (p. 12 in the appendix).

** The value of α_u corresponds to a value of Γ which is twice that determined by the formulas (83) to (86) in Chapt. X.

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The calculated day value of the attenuation constant α_u is a little greater in winter than in summer. The reason is that we have assumed the stellar ionization to penetrate to the same depth as the solar summer midday ionization. The lowest part of the stellar ionization will therefore in winter be lower than the lowest part of the solar midday ionization. But the stellar ionization is so small that it does not appreciably alter the ray paths, it simply increases the attenuation. It is possible that the penetrating power of the star light may be greater, and considerably greater, than that of sunlight. The highly penetrating ionization may even be a part of the stellar ionization and stellar radiation may be so constituted that the downward bend at an altitude of about 85 km in the ion-density curve (see Figs. IX, 6 and 7 in the appendix) disappears. Such an alteration in our assumptions would not vitiate our arguments, it would only bring the calculated ratio of the winter and summer midday intensities to a somewhat lower value.

From a comparison between Figs. IX, 10 and 12 (in the appendix), showing σ and $\Delta\varepsilon$ for summer time and the corresponding (not reproduced) figures for winter day time it appears that the influence of the earth's magnetic field on the value of the conductivity and the dielectric constant begins to manifest itself at an altitude two to three km higher in winter than in summer for long waves ($\lambda > 1000$ m). From Figs. IX. 13 to 18 it appears, however, that equal values of the refractive index are found at altitudes from four to five km higher in winter than in summer. Waves greater than 10000 m do not reach the magnetically influenced region on summer days, while these waves on winter days penetrate this region to a height of about two km. The downcoming waves will therefore in summer days keep their plane of polarization in the normal position, i. e. with the electric field vertical. Such waves therefore give true bearings on a loop aerial on summer days, see sect. 2 (f) above. On winter days the plane of polarization may be rotated somewhat and the bearings show variable and important deviations. At night these deviations become still greater. The above refers to distances less than 2000 km; for long distances where practically the whole radiation energy is carried by rays having very small earth angles the deviations from true bearings will always be small. All this is in good agreement with experience[1].

Somewhat shorter waves ($\lambda < 5000$ m) will reach the magnetic region even on summer days and will therefore show deviations in all cases in which the ground wave is not considerably stronger than the down-coming waves. But all down-coming rays with great earth angles will be greatly attenuated at daytime for all wave lengths between 5000 m and 400 m. Such waves therefore give only small deviations from true bearings during the daylight period. At night, however, the attenuation of the down-coming rays is small and the variations in bearings therefore are great. This is also in agreement with experience[2].

In addition these facts put rather narrow limits to the altitude of the lower boundary of the solar ionization for any atmosphere, such as F'. Thus if this altitude were reduced 10 km, there would theoretically be no difference between deviations of long waves on summer and winter days. And if this altitude were increased by 10 km the long waves should show considerable deviations even on summer days.

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[1] *R. L. Smith-Rose:* Radio Research Board. Special Reports Nos. 1—4. (1923—26).

[2] *R. L. Smith-Rose:* l. c. *R. Mesny:* l. c.

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It appears from formula (62) that the attenuation constants α_e and α_u decrease with increasing wave length while α_i is proportional to the total conductivity G of the intermediate layer, and this conductivity increases somewhat with increasing wave length. For very long waves and very great distances α_i will be the main part of the total attenuation constant and it should be possible to get a reliable estimate of G by means of long distance measurements on very long waves ($\lambda > 30000$ m).

(d) Other Evidence concerning the Height of the »Reflecting Layer«.

From our point of view there is in most cases, with the exception of very long waves, very little reflection and what is generally called reflection is in fact mainly refraction. But it may often be convenient to retain the reflection terminology. By the height of the »reflecting layer« we shall understand the height of the apex of the ray path — or in other words, the altitude of the horizontal part of the ray path.

This height of reflection will vary with the time of day and with the seasons, but it will also vary very much with the wave length and with the angle of incidence, a fact which seems to have been overlooked by some authors.

We have already touched upon this question when we were discussing the propagation of short waves. The most reliable measurements of the height of reflection have however been made with long waves, and we shall discuss the results of these measurements here.

Three essentially different methods for the measurement of the height of reflection have been used; in all three the shortest great circle distance between the transmitting and the receiving stations must be known. The first method is based on a counting of the number of wave lengths along the whole path from transmitter to receiver of the down-coming ray, assuming the values of the wave lengths along the path to be known. The second depends upon a measurement of the time taken in transmitting a signal from one station to another, assuming the signal-velocity to be known. In the third method the earth angle of the down-coming ray is measured. The details of each method may vary widely, but it falls outside the scope of the present work to treat all these questions thoroughly. We shall therefore only give the results of the available measurements and add, now and then, a few remarks.

In Fig. 24 we have shown the results of the different determinations which have been made up to now and in order to facilitate a comparison of these results with the requirements of the present theory and with the points of view of other authors, we have also included some further information[1].

In column I of Fig. 24 we have shown the entire possible range of heights of reflection (h_R) for all wave lengths smaller than 20000 m, for summer (S) and winter (W), day (D) and night (N). The values of h_R are taken from Figs. IX. 13 to 18 in the appendix.

For the very long waves the height of reflection is below 100 km, and there is very little difference between day and night, but about 5 km difference between summer and winter. The short waves penetrate to greater heights,

[1] It has not been possible to include in this figure the values obtained by *Appleton* and *Ratcliffe* by means of the earth angle method and published in the July No. of the Proc. Roy. Soc. (A), Vol. 115, p. 291, 305. 1927.

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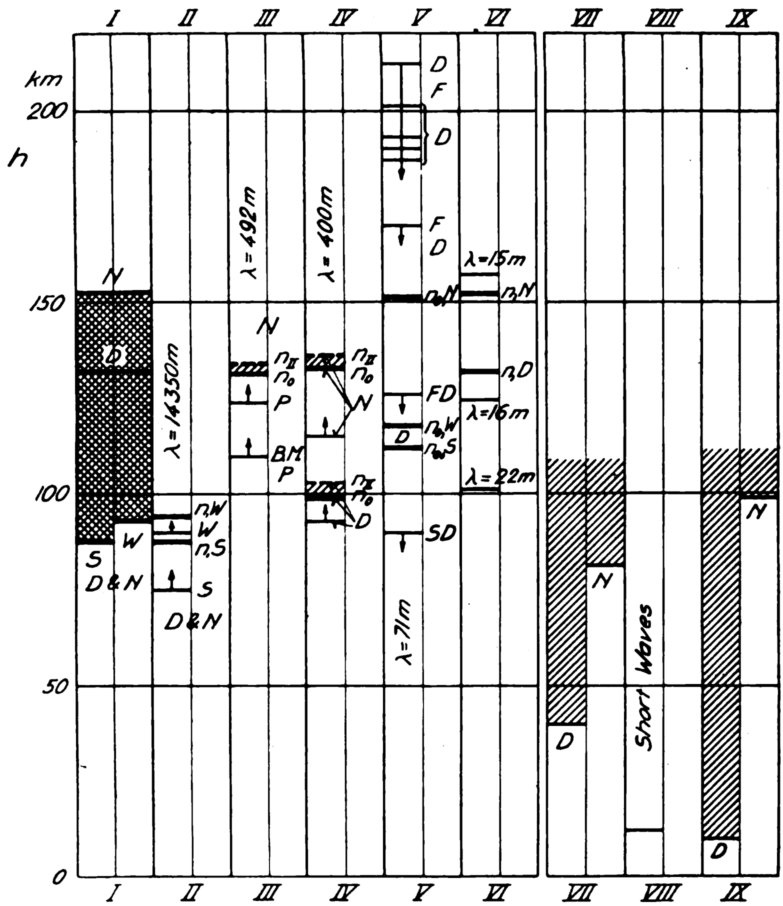

Fig. XI. 24. Column I shows the Range of the values of the Height of Reflection h_R according to Figs. IX, 13 to 18 (in the appendix).

Column II shows the values of h_R found by *J. Hollingworth*, Journ. Inst. E. Eng. Vol. 64, p. 586. 1926. ($\lambda = 14\,350$ m).

Column III shows the values of h_R found by *R. Bown*, *De Loss K. Martin* and *R. K. Potter*, Proc. Inst. Rad. Eng. Vol. 14, p. 87. 1926. ($\lambda = 492$ m).

Column IV shows the values of h_R found by *E. V. Appleton* and *M. A. F. Barnett*, Proc. Roy. Soc. (A). Vol. 113, p. 456. 1926. (See also Proc. Roy. Soc. Vol. 109, p. 621. 1925).

Column V shows the values of h_R found by *G. Breit* and *M. A. Tuve*, Phys. Rev. (II) Vol. 28, p. 568. 1926. ($\lambda = 71$ m).

Column VI shows the values of h_R from Table 9 for $\psi = 5^\circ$.

For Column VII see *T. L. Eckersley*, Proc. Wireless Sect. (Inst. El. Eng.). Vol. 2, p. 85—128. June 1927.

For Column VIII see *P. C. Oscanyan*, Proc. Inst. Rad. Eng. Vol. 15, p. 428. May 1927.

For Column IX see *J. H. Dellinger*, *L. E. Whittemore* and *S. Kruse*, Sc. Papers Bur. Stand. No. 476. 1923. (Fig. 9).

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especially by night, the shortest down-coming waves going up to 152 km. At these high altitudes there is, however, very little difference between summer and winter.

In columns II, III and IV we have shown the results of *Hollingworth*'s, *Bown*, *Martin* and *Potter*'s, and *Appleton* and *Barnett*'s measurements after the wave-length-number method. In details the three sets of measurements differ greatly from one another, but from a theoretical point of view they are all more or less alike. In regard to the theory we refer to Chapt. X, sect. 4.

The difficult point here is the question of the value of the wave length along the curved part of the path. This question is complicated very much by the earth's magnetic field. Any ray at any point where the magnetic field exerts influence will split up into four different rays corresponding to the four different refractive indices n_0, n_{I}, n_{II} and $n_{\perp}$. Any single ray leaving the transmitting station will give a beam of down-coming rays spreading over some distance. Since the attenuation of the rays »I« and »II« (see Chapt. VIII, sects. 2 and 4) is different and since the path of the rays I and II will also in general be different these down-coming rays will generally be elliptically polarized. And conversely at any receiving point there will be an infinite number of down-coming, differently polarized rays which are due to rays that have left the transmitting station at different earth angles.

The theory of this problem has not yet been worked out and it is difficult to judge to what extent the phase and the state of polarization of the received, down-coming rays are altered by the magnetic field. There is therefore some uncertainty regarding the numerical results obtained by this method.

In all three series, II, III and IV, the authors in calculating the height of reflection have simply assumed the wave length to be equal to λ_0, the wave length at the earth's surface. All three series calculated in this way give values of the height of reflection, which according to Chapt. X, sect. 4, are a little too small. The lines showing the measured values in II, III and IV are therefore provided with small arrows, pointing upwards. The corresponding values of $h_{0\mathrm{R}}$ and h_{IIR} calculated on the basis of the (n, h)-curves in Figs. IX 14 to 17 (in the appendix) are marked n_0, n_{II} and n; this last symbol is used when there is no appreciable difference between the values of h_{R} corresponding to n_0 and to n_{II}. Further particulars are to be found in Fig. 24 and in the text pertaining to this figure.

In column III, referring to the measurements of *Bown*, *Martin* and *Potter*, the line marked *B. M. P.* represents the value given by these authors, while the line marked *P* represents a recalculated value of $h_{0\mathrm{R}}$, which in our opinion agrees as well with the measurements as the value chosen by *Bown*, *Martin* and *Potter*. We do not, however, wish to insist upon this point.

We believe that at present this method of measuring the height of reflection is the most accurate, notwithstanding the above mentioned deficiencies, and of the three sets of measurements we place the greatest weight on the values found by *Appleton* and *Barnett*. Least reliable are perhaps *Hollingworth*'s results. But especially on the theoretical side work has still to be done before the height of reflection can with certainty be measured with an error not exceeding 5 km. In view of the splendid work already done, this goal will no doubt be reached.

The results of two sets of measurements after the method of time-lag in signal-transmission are given in columns V and VI. The first shows the values found by *Breit* and *Tuve* by nearly vertical reflection of a 71 m wave. The

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values marked S refer to summer, and those marked F to values found in the fall. The summer values are less than half the fall values, and the fall values differ very much among themselves. Since the summer values refer to July 28 and the fall values to Sept. 21—25, for which two seasons Figs. 25 shows the direction of the sun rays at noon at the place of observation, such a great difference between the summer and fall values is hardly explicable if the sun is the main source of ionization — which it undoubtedly is. Beautiful as the method of *Breit* and *Tuve* is, we believe that it lacks somewhat in accuracy [1].

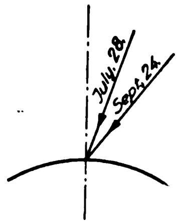

Fig. XI. 25. Direction of the Rays from the Sun in Washington July 28 and Sept. 24, referring to the Symbols S and F respectively in Column V of Fig. 24.

The values of *Breit* and *Tuve* are calculated by taking the signal-velocity equal to the velocity of light c. However, along the curved path the signal-velocity, as pointed out by *Breit* and *Tuve* [2], is less than c. The calculated values are therefore too high and the lines representing these values are provided with down-pointing arrows. In this case the conditions are also complicated by the influence of the magnetic field, but this influence is rather small for the short waves used by *Breit* and *Tuve*.

Column VI shows the results of the time-lag measurements on the direct and echo waves summarized in Table 9 and for $\psi = 5^0$ together with the day (n, D) and night (n, N) values for short waves determined on the basis of n-curves in Figs. IX. 13 to 17 (in the appendix). The experimental values ought to fall somewhere between the two last mentioned values. The accuracy of this method of measurement cannot be very great and there is also some uncertainty in regard to the ray path and the exact value of the signal velocity. But on the whole the experimental values agree rather well with our calculated values.

Considering the whole evidence of columns II to VI we believe that we are justified in stating, that the differences between the measured values of the height of reflection and our calculated values may be explained by the experimental errors and by transient irregularities in the normal state of ionization.

The earth angle method of measuring the height of the reflective layer has also some serious difficulties of which we shall mention a few. By this method it is the top C' of the equilateral triangle $AC'B$ (see Fig. X, 15) and not the apex c of the ray path which is determined. And it is assumed that at any instant there is only one down-coming ray, but for all waves, except the very long ones, there will, on account of the earth's magnetic field, be an infinite number of down-coming rays at any point. Besides there may possibly be rays which have made a different number of steps between the transmitting station and the receiver. This last case is illustrated by Fig. 22. At the receiving station B there are three down-coming rays: the ray 0 covering the whole distance AB in one step, the ray 1 taking two and the ray 2 taking six steps. But the existence of more than one down-coming ray makes the experimental determination of the earth angle rather uncertain even if the extra ray or rays

[1] *Breit* and *Tuve*, l. c., p. 569 point out some of the difficulties met with in the application of this method.

[2] See also Chapt. X, sect. 3.

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have smaller amplitude. We shall prove this for one of the experimental methods used by *E. V. Appleton* and *J. A. Ratcliffe*[1].

We assume the amplitudes E_1 and E_2 of two down-coming rays to be much smaller than the amplitude E_0 of the ground ray. It is easily proved[2] that the amplitude E_A of the electromotive force in a vertical aerial standing on highly conducting ground is determined by

$$\left.\begin{aligned} E_A = k_1\sqrt{E_0^2 + 4E_1^2\cos^2\psi_1 + 4E_2^2\cos^2\psi_2 + 4E_0E_1\cos\psi_1\cdot\cos\theta_1}\\ \overline{+ 4E_0E_2\cos\psi_2\cdot\cos\theta_2 + 8E_1E_2\cos\psi_1\cdot\cos\psi_2\cdot\cos(\theta_1-\theta_2)}, \end{aligned}\right\} \tag{68}$$

while the electromotive force E_L developed in a loop in the plane of propagation is determined by

$$E_L = k_2\sqrt{E_0^2 + 4E_1^2 + 4E_2^2 + 4E_0E_1\cos\theta_1 + 4E_0E_2\cos\theta_2 + 8E_1E_2\cos(\theta_1-\theta_2)}, \tag{69}$$

where k_1 and k_2 are constants the ratio of which may be determined experitally. ψ_1 and ψ_2 are the earth angles and θ_1 and θ_2 the phase angles of the down-coming rays.

Assuming $E_1^2 \ll E_0^2$ and $E_2^2 \ll E_0^2$ the formulas (68) and (69) reduce to

$$E_A \simeq k_1(E_0 + 2E_1\cos\psi_1\cdot\cos\theta_1 + 2E_2\cos\psi_2\cdot\cos\theta_2) \tag{70}$$

and

$$E_L \simeq k_2(E_0 + 2E_1\cos\theta_1 + 2E_2\cos\theta_2). \tag{71}$$

The ratio of the small variations $\varDelta E_A$ and $\varDelta E_L$ due to changes of intensity and of phase of the down-coming waves is then determined by

$$\frac{\varDelta E_A}{\varDelta E_L} = \frac{\cos\psi_1[\varDelta E_1\cdot\cos\theta_1 + E_1\cdot\varDelta(\cos\theta_1)] + \cos\psi_2[\varDelta E_2\cdot\cos\theta_2 + E_2\cdot\varDelta(\cos\theta_2)]}{[\varDelta E_1\cdot\cos\theta_1 + E_1\cdot\varDelta(\cos\theta_1)] + [\varDelta E_2\cdot\cos\theta_2 + E_2\cdot\varDelta(\cos\theta_2)]}\cdot\frac{k_1}{k_2}. \tag{72}$$

If

$$\varDelta E_2\cdot\cos\theta_2 + E_2\cdot\varDelta(\cos\theta_2) = 0 \tag{73}$$

we have

$$\frac{\varDelta E_A}{\varDelta E_L} = \frac{k_1}{k_2}\cdot\cos\psi_1, \tag{74}$$

and $\frac{k_1}{k_2}$ being known the value of $\cos\psi_1$ may be calculated from the ratio of simultaneous changes in E_A and E_L.

But if the equation (73) is not satisfied and if we still use equation (74) for the calculation of $\cos\psi$, the result may be somewhat erroneous even if E_2 is considerably smaller than E_1.

This method should therefore only be used in cases where it is quite certain that there is only one down-coming ray.

Besides the measurements already referred to in columns II—VI of Fig. 24, this figure also shows some hypothetical positions of the »reflecting layer« and corresponding states of ionization. Column VII shows the ionization assumed by *Eckersley*, going down to an altitude of about 40 km. Column VIII shows the height of a hypothetical reflecting layer for short waves assumed

[1] *E. V. Appleton* and *J. A. Ratcliffe:* Proc. Roy. Soc. (A). Vol. 115, p. 291, 305. 1927.

[2] Compare *Appleton* and *Ratcliffe:* l. c., p. 297.

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by *Oscanyan*, while column IX shows the state of ionization assumed by *Dellinger, Whittemore* and *Kruse* and mentioned previously.

We consider these three hypotheses as very improbable.

(e) Sunset and Sunrise Phenomena. Summer Night Intensity at Northern Latitudes.

Of the many peculiarities in radio wave transmission met especially at the time of sunset and sunrise we shall consider only those which — at least in our opinion — seem to be established without any doubt. For the present we shall base our discussion on the results obtained by *Lloyd Espenschied, C. N. Anderson* and *Austin Bailey*[1]. Figs. 26 and 27 are somewhat idealized reproductions of Figs. 8 and 6 of the paper cited.

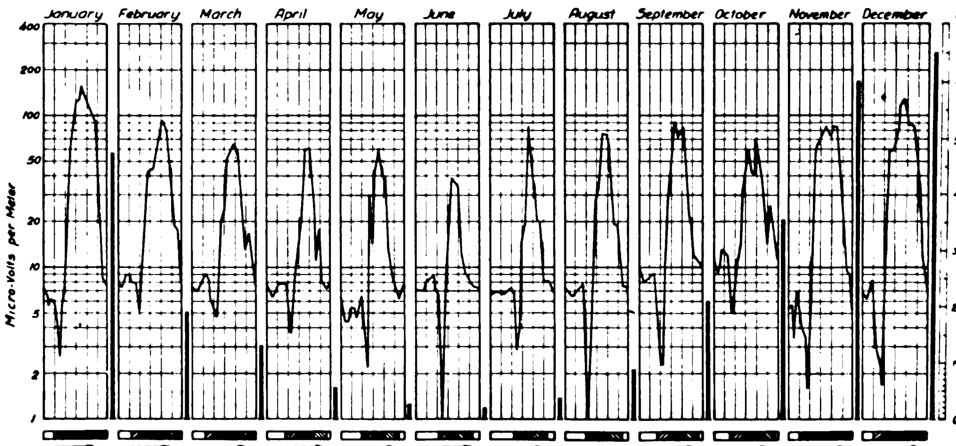

Fig. XI. 26. Monthly averages of diurnal variation in signal field at New Southgate, England. Transmission from Rocky Point L. I. Distance 5480 km. Frequency 57 000 cycles ($\lambda = 5270$ m). Antenna current 300 amp. (A somewhat idealized reproduction of Fig. 8 of *Espenschied, Anderson* and *Bailey*'s paper).

In the figures below the black portions indicate the time during which the trans-Atlantic path is entirely in darkness, the shaded portions the time during which it is only partially in darkness, and the unshaded portions the time during which daylight pervades the entire path.

The heavy lines between the different months show the lowest altitudes of direct sunlight at midnight at a northern latitude of $52\frac{1}{2}^0$ on the first day of the following month.

We shall first consider the reduced intensity during the summer nights. In Fig. 28 *SR* indicates sun rays, which at midnight at summer solstice (June 22) graze the earth at the arctic polar circle *A*. The point *B* has a northern latitude of about $52\frac{1}{2}^0$, and the lower boundary of the direct sunlight (ray 1) is at midnight at an altitude of 195 km. Of ray 1 all the ultra-violet radiation is absorbed before reaching the point *A*. This ray, therefore, has no influence on the ionization in the atmosphere south of *A*. Ray 2, which at *A* has an altitude of approximately 110 km, will here still have some of its ultra-violet

[1] *Lloyd Espenschied, C. N. Anderson* and *Austin Bailey:* Proc. Inst. Rad. Eng. Vol. 14, p. 7—56. 1926.

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radiation left. Some of this radiation will be scattered by the molecules of the air above B and part of this scattered radiation will penetrate down to an altitude of 90 to 100 km above B — indicated by p — and there cause ionization which will increase the attenuation of the waves. It must be remembered — compare Chapt. V sect. 11 – that the scattering effect is proportional to the fourth power of the frequency of the light scattered and therefore very effective for the ultra-violet radiation causing the ionization of the atmosphere. With our present scanty knowledge of the energy distribution of the ultra-violet radiation from the sun it will hardly be possible to make any reliable numerical estimate of this influence. But even a small increase in the electron density is sufficient to reduce the field strength of the received wave by two to three times, the amount here in question.

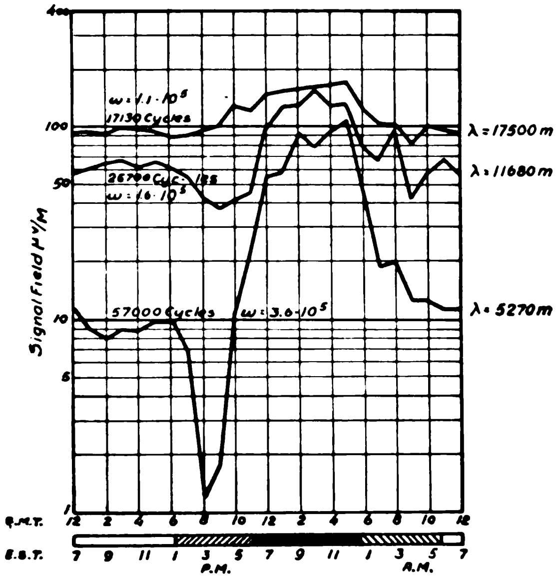

Fig. XI. 27. Monthly average of diurnal variation in signal field strength from American stations on various frequencies as received at New Southgate, England. September 1923.

The same influence will be at work for some time after sunset and before sunrise, and the high night intensities rise and fall gradually, giving an almost horizontal top for the night intensity in winter and a pointed top in summer (see Fig. 26).

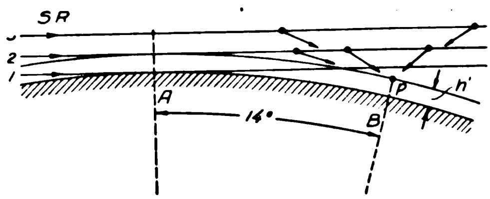

Fig. IX. 28. Midnight position of a point B of northern latitude $52\frac{1}{2}{}^0$ at Summer Solstice. A arctic circle.

Fig. 26 shows that for the 5270 m wave a decided drop in the received field strength accompanies the occurrence of sunset in the transmission path between transmitter and receiver, *i. e.* with day at the transmitter, night at the receiver. This figure also shows that such a drop in field intensity does not accompany the occurrence of sunrise in the transmission path, *i. e.* with night at the transmitter and day at the receiver.

In Fig. 29 we have shown a diagrammatical representation of transmission conditions in the two cases, Part I referring to transmission from day to night, Part II from night to day.

We shall first consider Part I. The direct ground wave leaving the transmitter T will not reach a receiver in the neighbourhood of B with any measurable intensity. Ray 1, leaving the transmitter tangentially to the earth's surface will reach the region at B with a considerable intensity but owing to the increasing altitude of the »reflecting layer« between b and c from the

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day value H_D to the night value H_N, this ray does not come down to the surface of the earth on the night side. The same is the case with ray 2 which leaves the transmitter at a finite earth angle. Ray 3, however, leaving the transmitter at an earth angle ψ_0 will just come down to the surface at the point B, and rays leaving at earth angles greater than ψ_0 will all come down on the night side as well as on the day side.

On the other hand, we have seen that the attenuation of the rays increases very rapidly with increasing earth angle and the field strength at B will therefore be considerably less than it would have been if the whole transmission path had been sunlit or had been wholly on the night side of the earth.

We shall next consider transmission from night to day. Part II of Fig. 29 shows that all rays leaving the transmitter T by night come down to the sur-

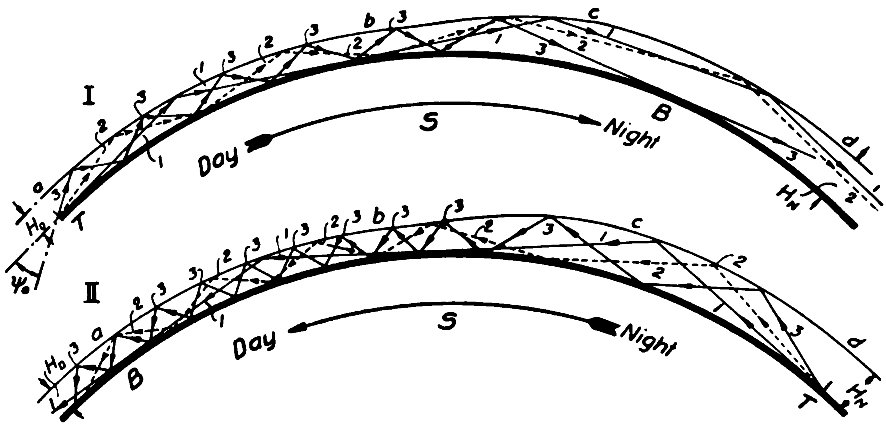

Fig. XI. 29. The heavy lines represent the surface of the earth, the lines *abcd* the effective »reflecting layer« for the wave length considered. (For practial reasons the height of this layer has been made too great; otherwise the space between the layer and the earth would be too crowded with ray paths).

face of the earth on the day side. In going from full night condition to full day condition the field strength should therefore fall off gradually. There may, however, be some small drop in intensity below full daylight values on account of increased attenuation on the day side caused by the increase in the earth angles of the rays.

In both cases, I and II, rapid variations in field intensities may take place on account of alterations in phase between the different rays.

For very long waves ($\lambda > 18000$ m) the sunset drop found in case I for the 5270 m wave must disappear, since the height of the »reflecting layer« is almost the same by day and by night $H_D \sim H_N$ (see Chapt. IX, Figs. 13 to 17). This is in complete agreement with the results of *Espenschied, Anderson* and *Bailey* (see Fig. 27).

(f). Meteorological Influences on the Propagation of Radio Waves.

The influence of the meteorological conditions of the atmosphere on the intensity and bearing of long radio waves is generally rather small[1] even

[1] *R. L. Smith-Rose:* Radio Research Board. Special Report No. 3, p. 106. 1925.

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though cases are known in which thunder storms have caused very great deviations in the bearing of long waves[1]. We shall not enter into the discussion of this problem but only remark that in our opinion most of these effects are due to reflections at the surfaces of discontinuity so often found in the lower atmosphere. For the theory of these reflections we may refer to Chapt. VIII, sects. 3 and 4.

4. *Propagation of Waves of Medium Length ($\lambda \simeq 100$—400 m).*

In short wave transmission the whole radiation energy is carried by the free, down-coming rays while the ground ray is unimportant. Since the conductivity of the ionized regions of the upper atmosphere increases rapidly with increasing wave length, the range of these waves decreases with increasing wave length.

In typical long wave transmission the main part of the energy is carried by the ground wave and since the earth loss decreases rapidly with increasing wave length the range in long wave transmission will — at least up to a certain point — increase with increasing wave length.

The range must therefore have a minimum at some intermediate wave length. This is so far in agreement with experience as both day and night ranges have minima for wave lengths in the neighbourhood of 200 m. But the day range has extraordinary low values especially with wave lengths between 100 and 200 m. In Fig. 30 the curve *a b c d e* shows such a day range curve according to *Taylor* and *Hulburt*[2]. A curve of the type *a b c′ d e* was to be expected but there can be no doubt as to the reality of the extra dip down to *c*. There has been some uncertainty as to whether the night range curve also shows any deep depression in the vicinity of $\lambda = 200$ m. It has, however, been proved by *A. Meissner*[3] and *T. L. Eckersley*[4] that if such an effect exists, it is at least much less pronounced than for the day range.

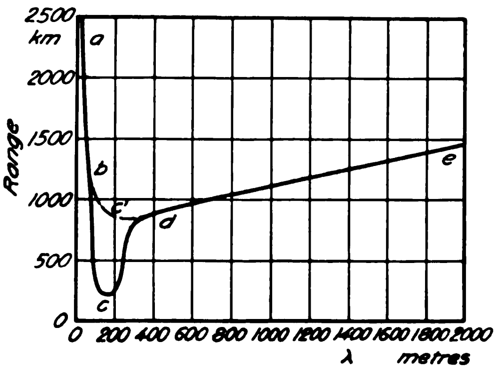

Fig. XI. 30. Day Range Curve according to *Taylor* and *Hulburt*.

Appleton and *Barnett*[5], *Nichols* and *Schelleng*[6], *Taylor* and *Hulburt*[7] and others have tried to explain this extra depression in the range values near $\lambda = 200$ m by the influence of the earth's magnetic field and have called attention to the fact that free electrons in a magnetic field of 0.5 gauss will be in »resonance« for a wave length of 214 m. Such electrons will therefore be set in very violent,

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[1] *M. Bäumler* and *J. Zenneck:* E. N. T. Vol. 3, p. 139—141. 1926.

[2] *A. Hoyt Taylor* and *E. O. Hulburt:* Phys. Rev. (II) Vol. 27, p. 194. 1926.

[3] *A. Meissner:* E. N. T. Vol. 3, p. 321—324. 1926.

[4] *T. L. Eckersley:* l. c., p. 114.

[5] *E. V. Appleton* and *M. A. F. Barnett:* »Electrician«, April 3rd, 1925; Proc. Camb. Phil. Soc. Vol. 22, Part 5, p. 672—675. 1925.

[6] *H. W. Nichols* and *J. C. Schelleng:* Bell System Techn. Journal. Vol. 4 No. 2. 1925.

[7] *A. Hoyt Taylor* and *E. O. Hulburt:* l. c.

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forced oscillation. By the (rare) collisions between the electrons and the molecules this energy of oscillation is partly converted to thermal energy, the result being a great attenuation of the wave. *A. Meissner*[1] has called the attention to the following difficulty in this explanation. The theory assumes that the time between two consecutive collisions of an electron is large in comparison with the period of the wave, and this condition is more nearly fulfilled by night than by day[2]. Therefore resonance attenuation should be more pronounced by night than by day, but the reverse is the case.

A final solution of this problem cannot be obtained until the theory of wave propagation in a non-homogeneous ionized atmosphere in a magnetic field has been further developed, but some information may be given with the material at hand.

From Figs. IX, 15 and 17 (in the appendix) we have for $\omega = 10^7$ determined the reciprocal of the values of the attenuation constants for the magnetically different rays $\gamma_0, \gamma_{\mathrm{I}}, \gamma_{\mathrm{II}}$ and $\gamma_{\perp}$ at the altitudes at which the refractive index is equal to 0.95, 0.9 and 0.8 respectively. These values for day and night, summer and winter, have been collected in Table 11, which also contains the earth angles $\psi_{\max}$ for rays having their apex at the altitudes at which the refractive index has the specified values.

It appears from this table that the day values of γ_{I} and $\gamma_{\perp}$ are very high, while the night values generally are more than ten times smaller and of the same order of magnitude as the day values without any magnetic field.

Table 11. Values of $\frac{1}{\gamma}$ at the Apex of the Path for the specified values $\psi_{\max}$ of the Earth Angles and for $\omega = 10^7$.

n	$\psi_{\max}$	Season	$\frac{1}{\gamma_0}$ Day	$\frac{1}{\gamma_0}$ Night	$\frac{1}{\gamma_{\mathrm{I}}}$ Day	$\frac{1}{\gamma_{\mathrm{I}}}$ Night	$\frac{1}{\gamma_{\mathrm{II}}}$ Day	$\frac{1}{\gamma_{\mathrm{II}}}$ Night	$\frac{1}{\gamma_{\perp}}$ Day	$\frac{1}{\gamma_{\perp}}$ Night
			km	km	km	km	km	km	km	km
0.95	14° to 15°	Summer	60	2300	3.8	40	130	4300	4.5	70
		Winter	120	2000	6	40	280	5000	8	80
0.90	23° to 24°	Summer	35	1200	2	33	80	2200	2.5	90
		Winter	70	1000	4	36	150	2500	4.5	90
0.80	35° to 36°	Summer	20	560	1	40	50	1900	1.4	55
		Winter	35	550	2	40	90	1300	2.5	50

[1] *A. Meissner:* l. c.
[2] See also Chapter VII, p. 103—4.

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Even if it is not possible to say to just what extent the total radiated energy will be attenuated, it is evident that the effective day attenuation for the 188 m wave ($\omega = 10^7$) must be very great[1] and much greater than the night attenuation. Accordingly we believe that the extraordinary dip in the range curve for the wave lengths between 100 and 200 m is due mainly to the influence of the earth's magnetic field, but it is, as explained in Chapt. VII, hardly justifiable to speak of any resonance phenomenon.

For the present, we are content with the qualitative agreement between our theory and experience and shall not try to give an exact theory of the shape of the day range curve for wave lengths near the critical values, 100 to 200 m[2].

With regard to fading and deviation from true bearing, the medium waves present no special particularity, but only form a connecting link between the long and short waves.

---

[1] Since the energy of any ray along the curved part of the path will be continually changing over to the other kinds of rays the influence of the great attenuation constants, γ_{I} and $\gamma_{\perp}$, will predominate and it is of relatively small importance that one of the constants, γ_{II}, is rather small.

[2] The recent paper by *E. O. Hulburt* (Phys. Rev. (II), vol. 29, p. 106. May 1927) bearing on this question suffers from the inevitable difficulties mentioned above. In addition we do not agree with his assumptions regarding atmospheric ionization. The author's approximate values for γ_{I} and γ_{II} differ a little from ours. His value of $\gamma_{\perp}$ does not agree with ours.

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It has been possible to give a qualitative — and in many cases even a quantitative — explanation of all the well known facts of radio wave propagation. For the further promotion of the study and knowledge of this fascinating and important problem we are, however, in need of substantial development in two important fields of physics: (1) The theory of the propagation of electro-magnetic waves in an ionized, non-homogeneous atmosphere in a constant magnetic field. (2) The experimental knowledge of the ionic properties of gases especially in high vacuum: the lifetime of free electrons, the value of the recombination constant, the length of the free path of slow electrons especially in helium, etc.[1].

With such new knowledge it will, no doubt, be possible on the basis of the steadily increasing experimental data of radio transmission, and by means of the results to be obtained by the experimental methods of *Appleton*, *Breit* and *Tuve* and others, to give a quantitative account of all the main problems of radio wave propagation and thereby establish a secure foundation for the further development of the art of radio communication. We will, at the same time, get rather definite information about the constitution and pressure of the upper atmosphere, and much light will be thrown on geophysical and cosmic problems.

[1] It should be remembered, however, that the upper atmosphere may, after all, offer the only experimental possibility for the study and solution of some of the questions, since laboratory methods for the study of wave motion in an atmosphere containing electrons with free paths of 10 to 100 m are hardly feasible.

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UDVALGET FOR INGENIØRVIDENSKABELIG FORSKNING

THE PROPAGATION OF RADIO WAVES

ALONG THE SURFACE OF THE EARTH AND IN THE ATMOSPHERE

BY

P. O. PEDERSEN

COPENHAGEN

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For the atmospheres F and F' see Chapt. IV (Tables 2 and 3) and Chapt. IX (Table 7).

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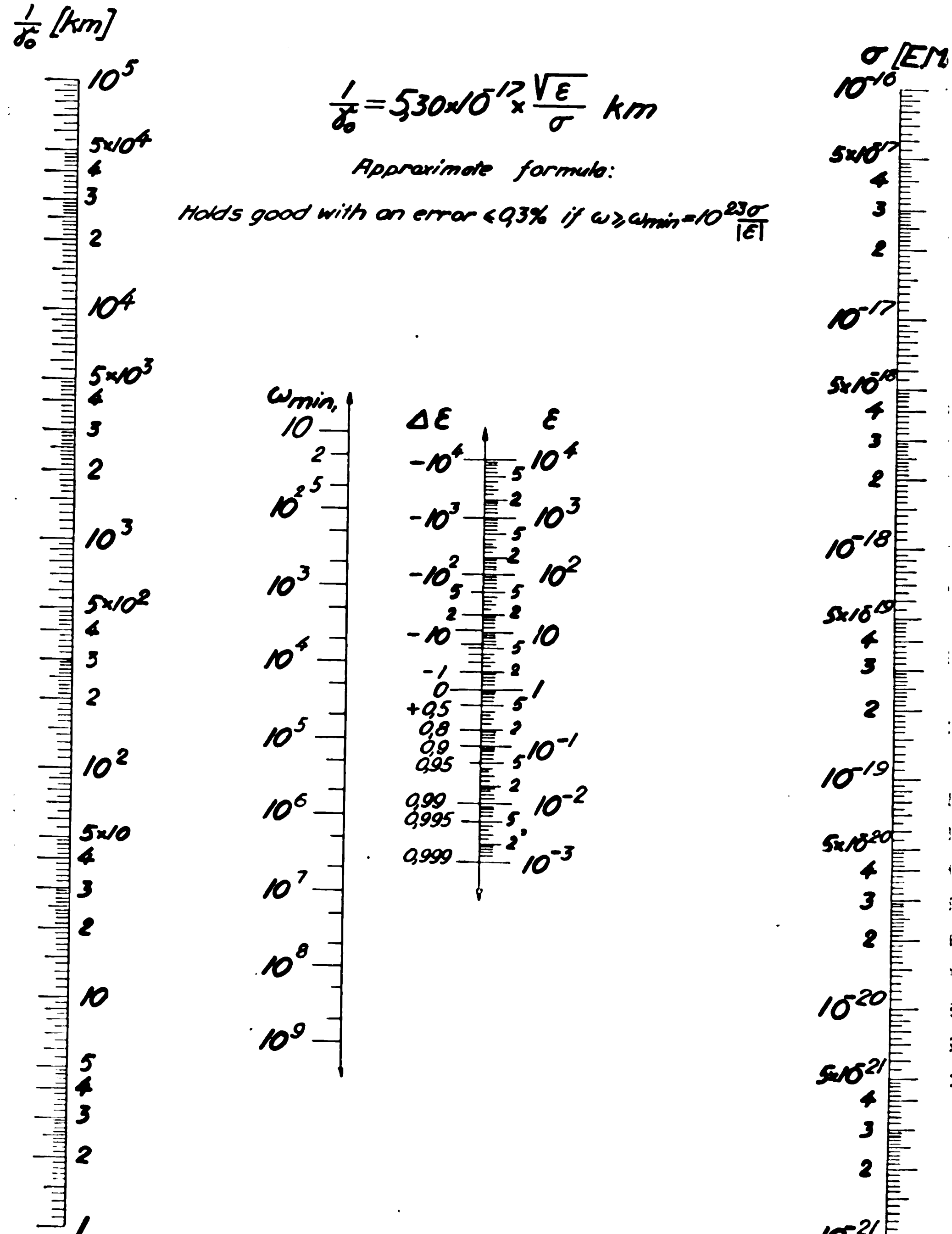

Fig. VIII. 4. Nomogram for $\frac{1}{\gamma_0}$. A straight line connecting the given values of $\Delta\varepsilon$ and σ determines the value of $\frac{1}{\gamma_0}$ on the basis of the approximate formula $\frac{1}{\gamma_0} = 5.3 \cdot 10^{-17} \frac{\sqrt{\varepsilon}}{\sigma}$ km. This line at the same time determines $\omega_{\min}$, i. e. the minimum value of ω for which the approximate formula holds good. If $\omega < \omega_{\min}$ the exact nomogram for $\frac{1}{\gamma_0}$ must be used (Fig. 5). The reading is most conveniently effected by using a fine scratch on the bottom side of a transparent rule, for instance a strip of celluloid.

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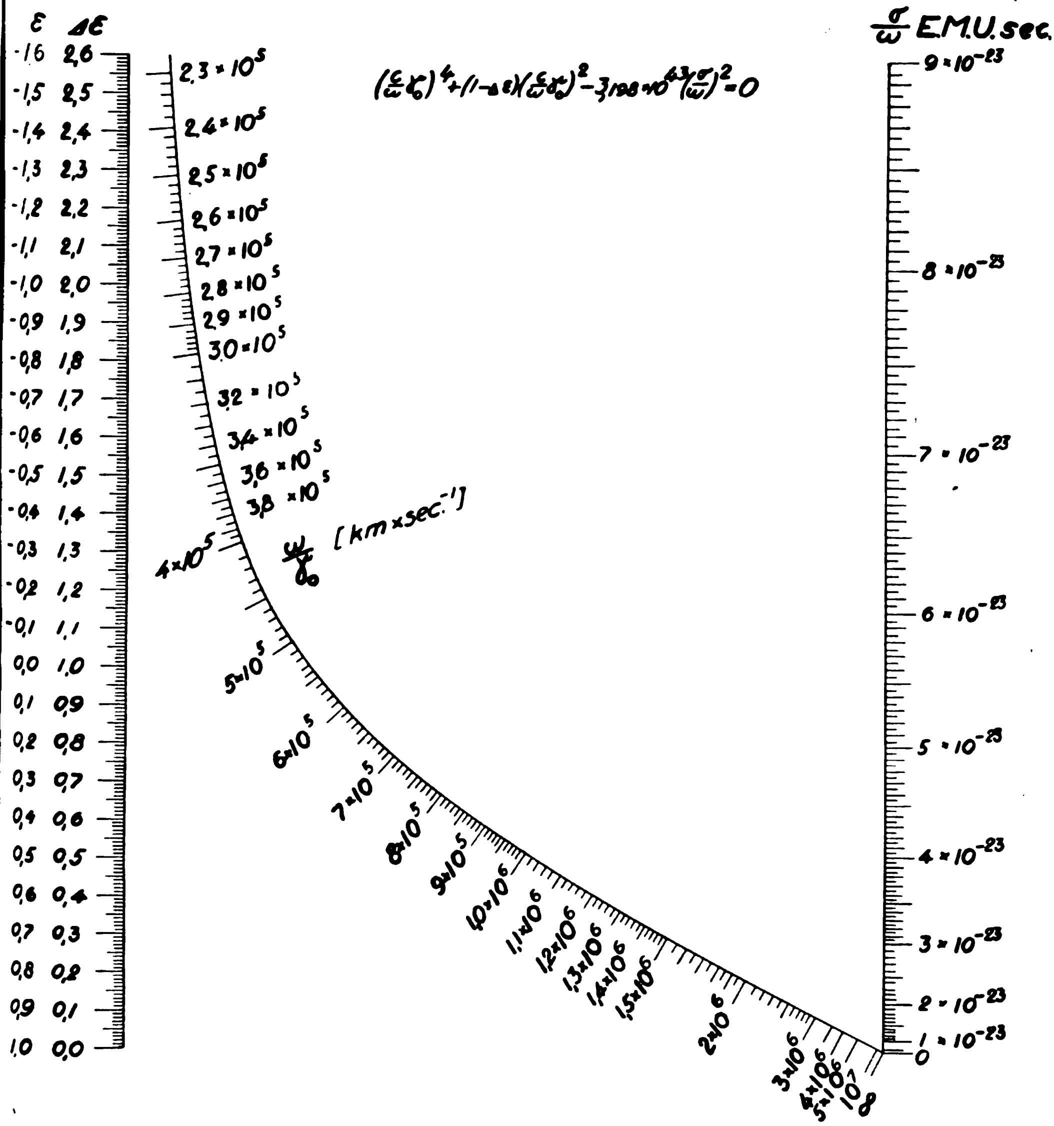

Fig. VIII. 5. Nomogram for $\frac{\omega}{\gamma_0}$. A straight line connecting the given values of ε or $\Delta\varepsilon$ and $\frac{\sigma}{\omega}$ determines $\frac{\omega}{\gamma_0}$ which latter when divided by ω gives $\frac{1}{\gamma_0}$ in km. If the given values fall outside the diagram, they may be replaced by $\varepsilon_1 = \frac{1}{a}\,\varepsilon \left(\text{i. e. } \Delta\varepsilon_1 = 1 - \frac{1}{a} + \frac{\Delta\varepsilon}{a}\right)$ and $\frac{\sigma_1}{\omega} = \frac{1}{a} \cdot \frac{\sigma}{\omega}$, and the resulting quantity will then be $\frac{\omega}{\gamma_0} \cdot \sqrt{a}$. In other words, the value of $\frac{\omega}{\gamma_0}$ thus found has to be divided by $\sqrt{a}$.

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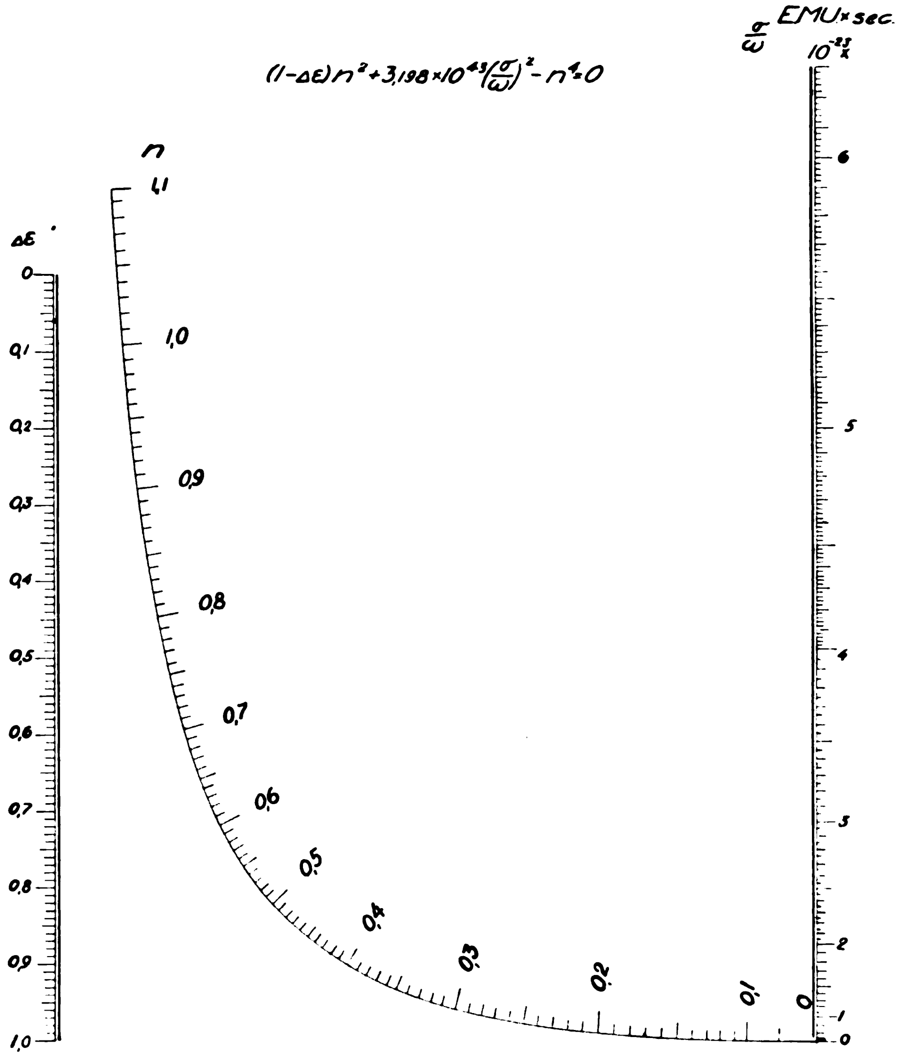

Fig. VIII. 7. Nomogram for n (exact). A straight line connecting the points of the scales of $\Delta\varepsilon$ and $\frac{\sigma}{\omega}$ corresponding to the given values of these quantities will intersect the scale of n at the desired value of n. If $\frac{\sigma}{\omega}$ is too large, $\frac{\sigma_1}{\omega} = \frac{1}{a} \cdot \frac{\sigma}{\omega}$ and $\varepsilon_1 = \frac{1}{a} \cdot \varepsilon \left(\text{i. e. } \Delta\varepsilon_1 = 1 - \frac{1}{a} + \frac{\Delta\varepsilon}{a}\right)$ may be used instead, and the value found for n then has to be multiplied by $\sqrt{a}$.

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$$n_{\text{app}} = 5.65 \cdot 10^{-21} \frac{\sigma}{\omega} \frac{1}{\sqrt{|\varepsilon|}}$$

$$\left[n^4 + (\Delta\varepsilon - 1) n^2 - 3.198 \cdot 10^{43} \left(\frac{\sigma}{\omega}\right)^2 = 0 \right]$$

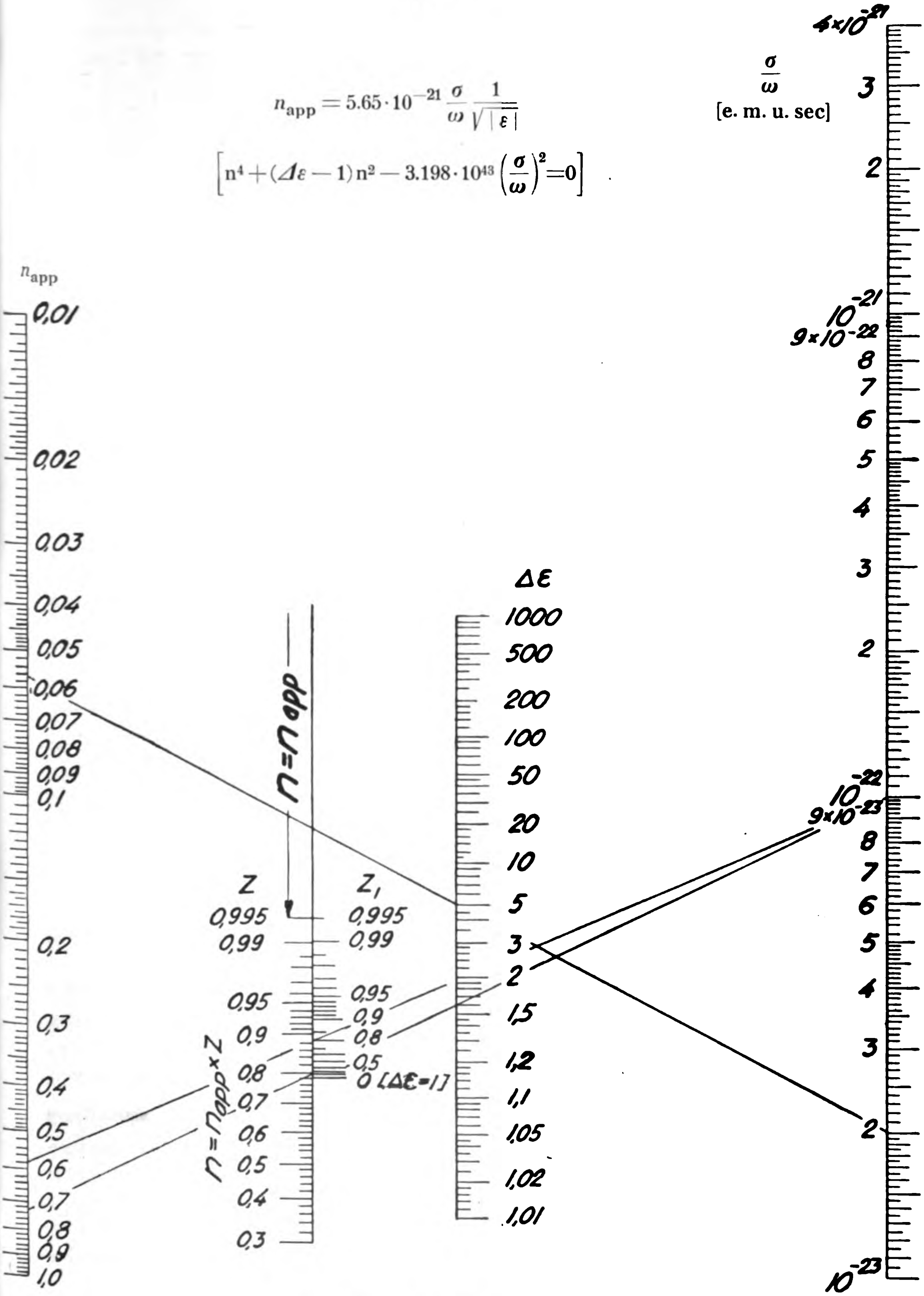

Fig. VIII. 8. For text see next page.

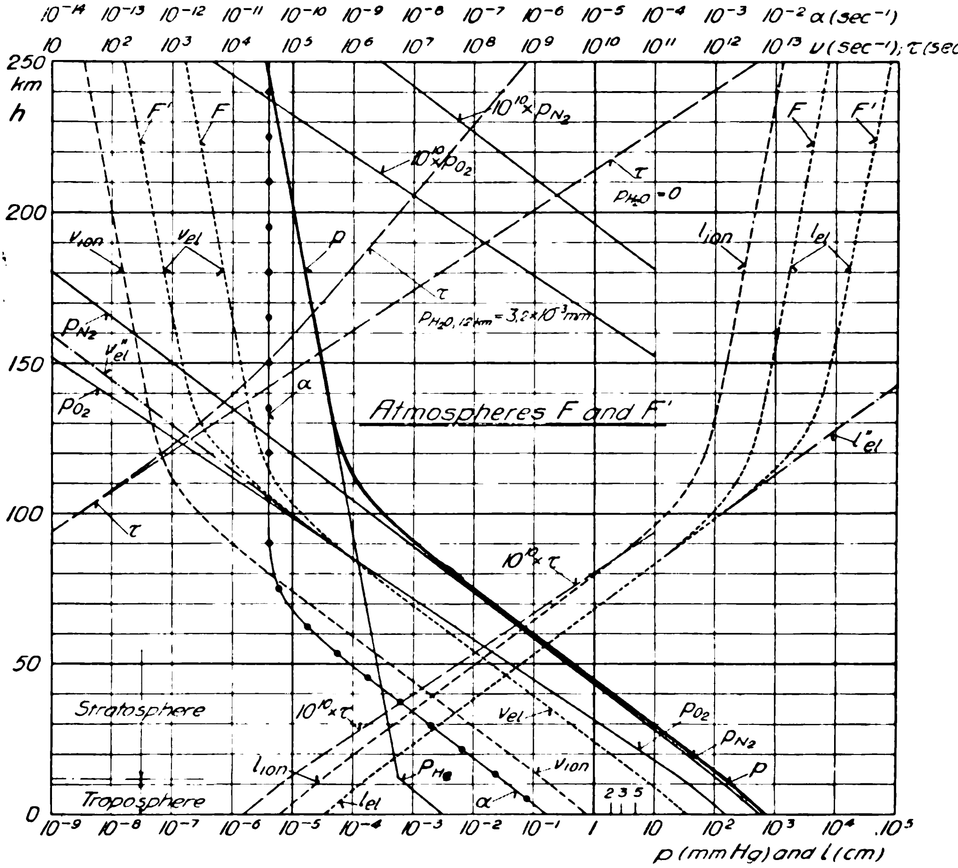

Fig. IX. 1. The total pressure p, the partial pressures p_{N_2}, p_{O_2} and p_{He}, the mean free paths l_{el} and l_{ion}, the collision numbers ν_{el} and ν_{ion}, and the recombination constant α as functions of the altitude h for the »atmospheres« F and F'. (See Chapts. IV and IX).

Fig. VIII. 8. Nomogram for n. A straight line connecting the given values of $\Delta\varepsilon$ and $\frac{\sigma}{\omega}$ determines n_{app}. If the said line intersects the scale of z above the point $z = 0.995$, n will be equal to n_{app}. If the point of intersection is below $z = 0.995$, n will be equal to $z \cdot n_{app}$. This multiplication may be performed by swinging the straight line, about the point $\frac{\sigma}{\omega}$, from the point just found on the scale of z to the correspondingly marked point on the scale of z_1. The new position of the straight line will then intersect the scale of n_{app} at a point indicating the exact value of n. Numerical examples: $\frac{\sigma}{\omega} = 2 \cdot 10^{-23}$ for $\Delta\varepsilon = 5$ gives $n = n_{app} = 0.0565$; $\frac{\sigma}{\omega} = 10^{-22}$ and $\Delta\varepsilon = 1.6$ correspond to $n_{app} = 0.73$ and $z = 0.80$, and then $\frac{\sigma}{\omega} = 10^{-22}$ and $z_1 = 0.80$ determine the final value $n = 0.580$.

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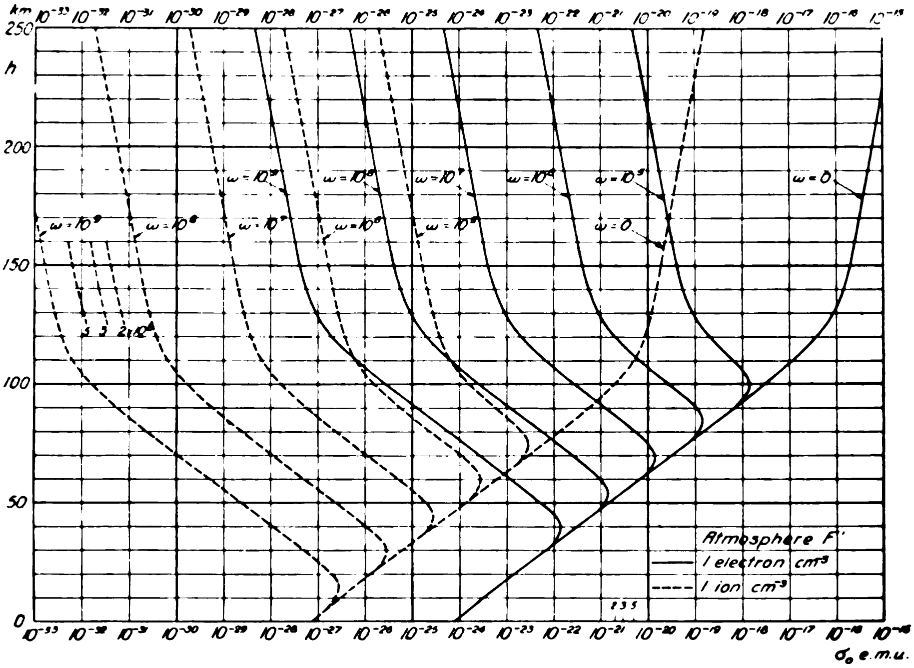

Fig. IX. 2. The conductivity σ_0 caused by one electron or one ion per c c as a function of the altitude and of the frequency for the atmosphere F'.

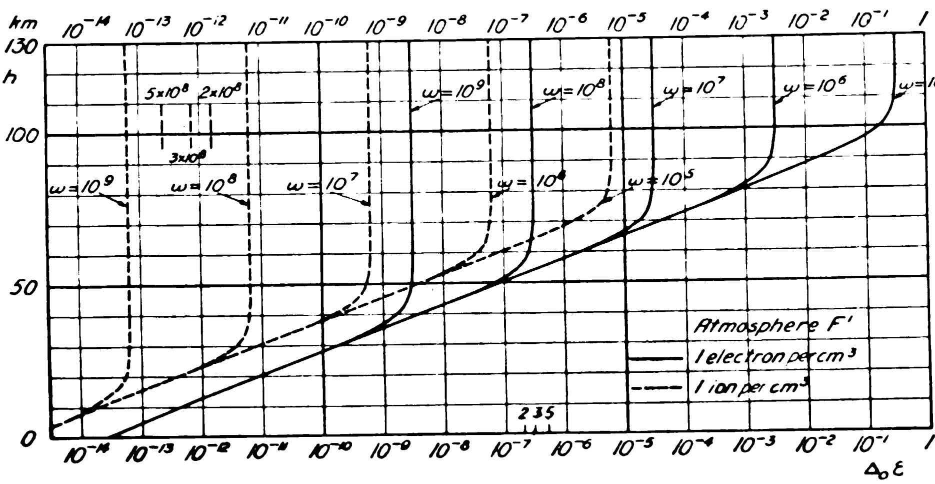

Fig. IX. 4. The reduction in the dielectric constant caused by one electron or one ion per c c as a function of the altitude and of the frequency for the atmosphere F'.

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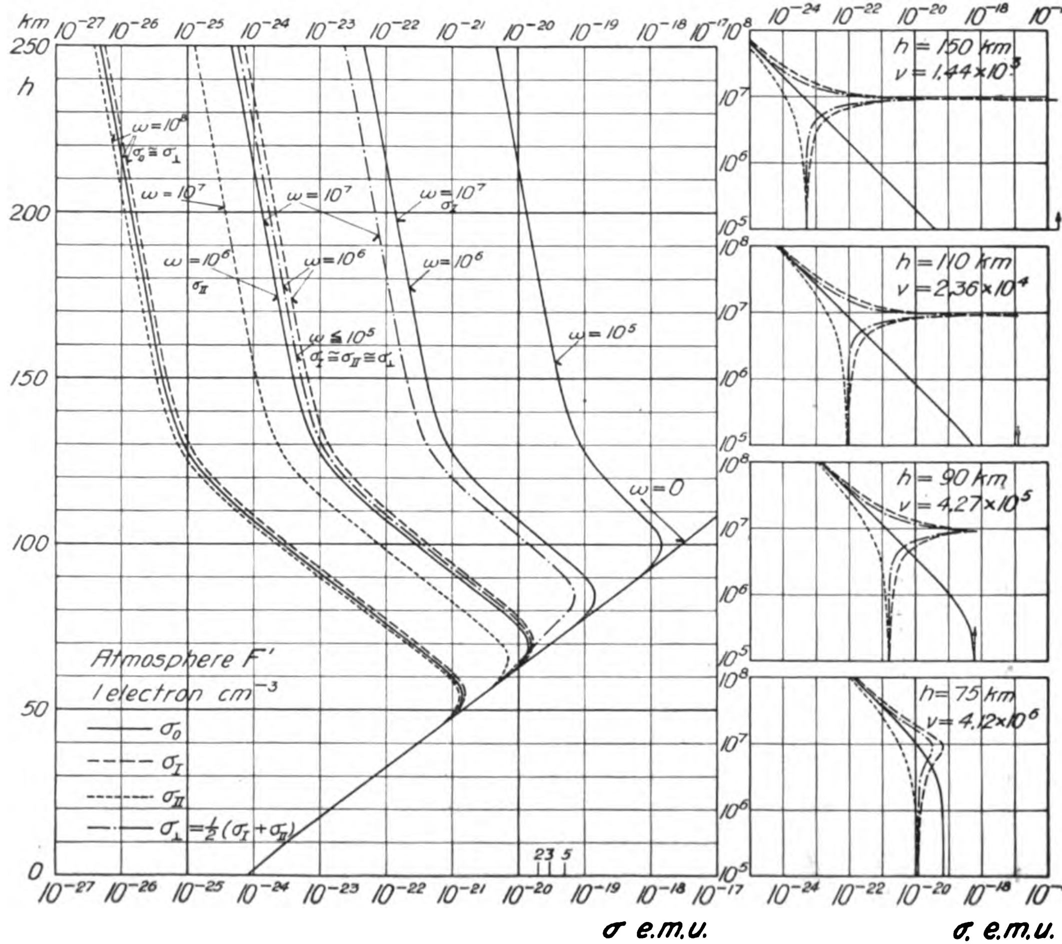

Fig. IX. 3. The conductivities σ_0, σ_I, σ_{II} and $\sigma_\perp$ caused by one electron per c c as functions of the altitude and the frequency for the atmosphere F'. In calculating the values of σ_I, σ_{II} and $\sigma_\perp$ it is assumed that the intensity of the magnetic field is about H = 0.51 gauss, corresponding to $h = \frac{e}{mc} \cdot H = 9 \cdot 10^6$ (see p. 98 and 103).

The right hand part of the figure shows σ_0, σ_I, σ_{II} and $\sigma_\perp$ as functions of the frequency at the altitudes 150, 110, 90 and 75 km.

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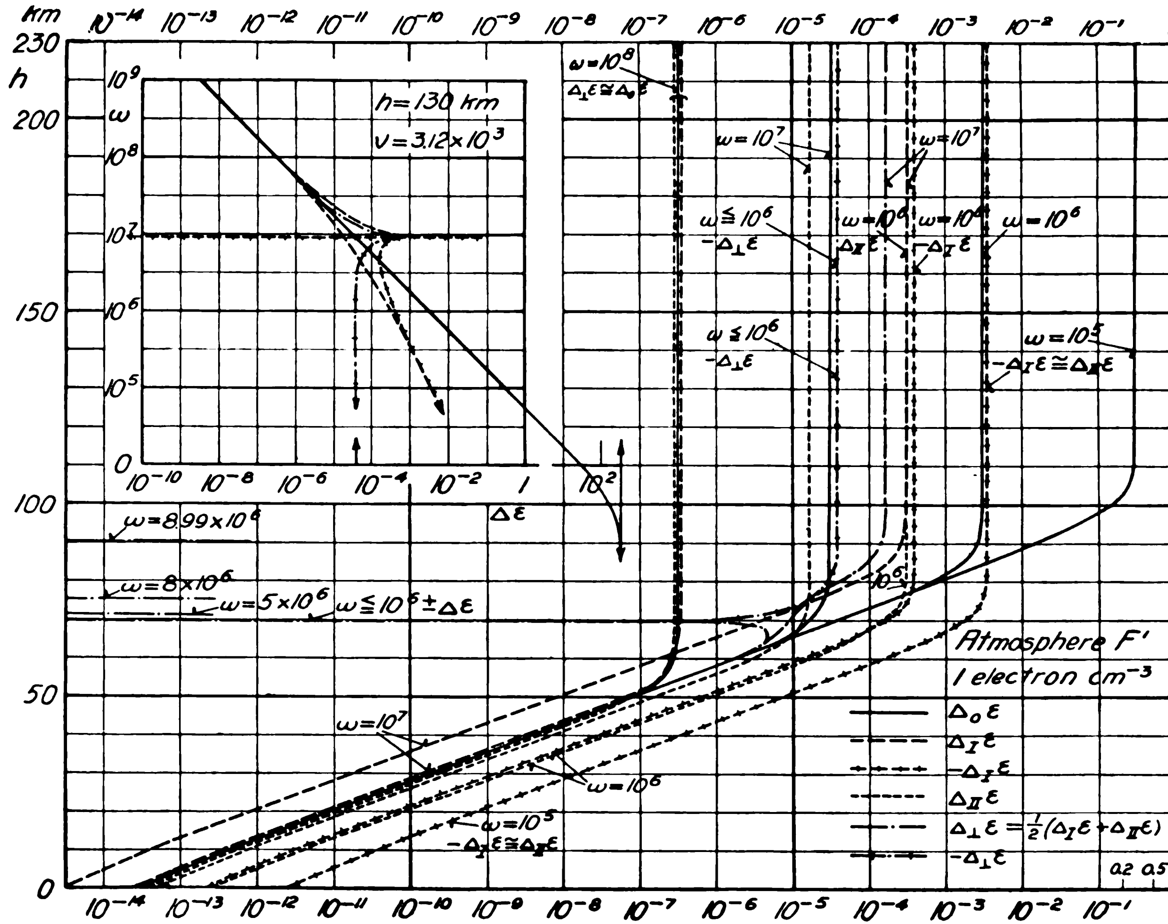

Fig. IX. 5. The reductions $\Delta_0\varepsilon$, $\Delta_{\mathrm{I}}\varepsilon$, $\Delta_{\mathrm{II}}\varepsilon$ and $\Delta_{\perp}\varepsilon$ in the dielectric constant caused by one electron per cc as functions of the altitude and the frequency for the atmosphere F'. (Magnetic field $\mathrm{H} = 0.51$ gauss; $h = 9 \cdot 10^6$).

The figure in the upper left hand corner shows the values of $\Delta_0\varepsilon$, $\Delta_{\mathrm{I}}\varepsilon$, $\Delta_{\mathrm{II}}\varepsilon$ and $\Delta_{\perp}\varepsilon$ as functions of the frequency at an altitude of 130 km.

Chapter VII Figs. 10 and 11 show the same quantities at the altitudes 40 km and 80 km for the atmosphere F. (For these low altitudes the difference between the atmospheres F and F' may be neglected).

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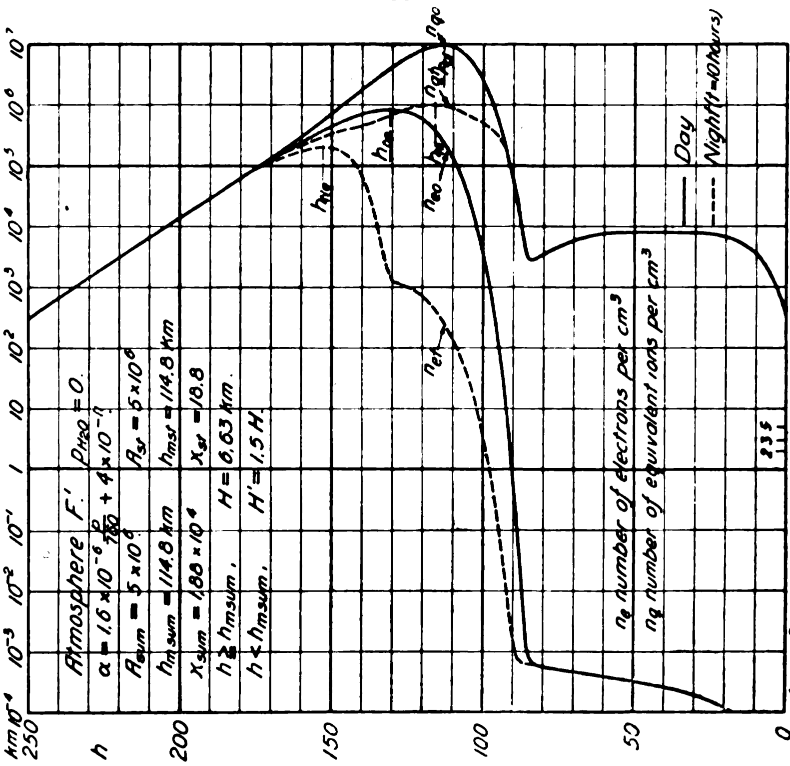

Fig. IX. 6. Number of electrons (n_e) and of equivalent ions (n_q) per c c as a function of the altitude. *F'*. Summer.

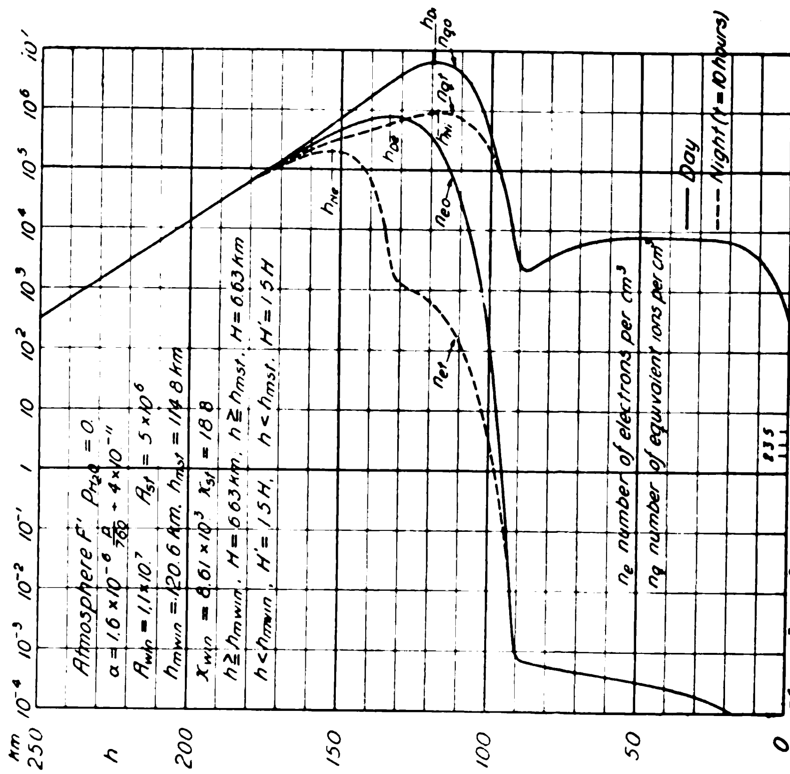

Fig. IX. 7. Number of electrons (n_e) and of equivalent ions (n_q) per c c as a function of the altitude. *F'*. Winter.

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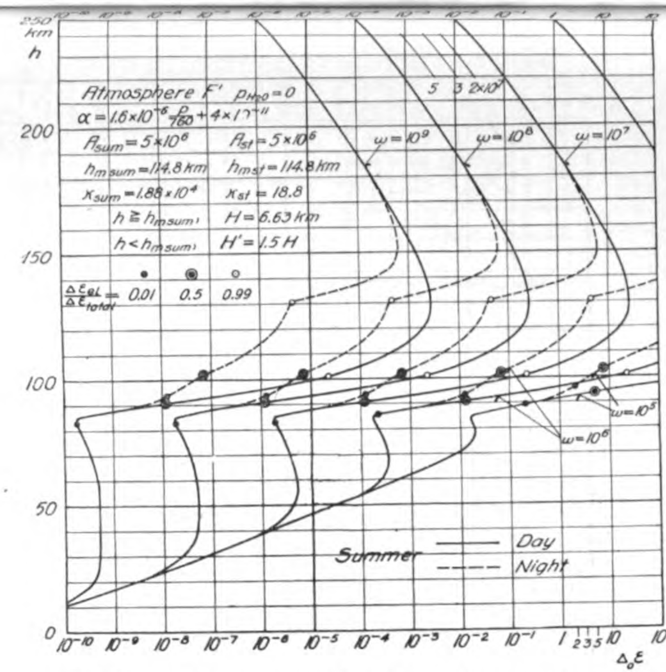

Fig. IX. 9. $\Delta_0\varepsilon$ as a function of h and ω from $h = 0$ to $h = 250$ km. The values correspond to the constants given in table 7 (Chapt. IX) for the atmosphere F'. Summer time.

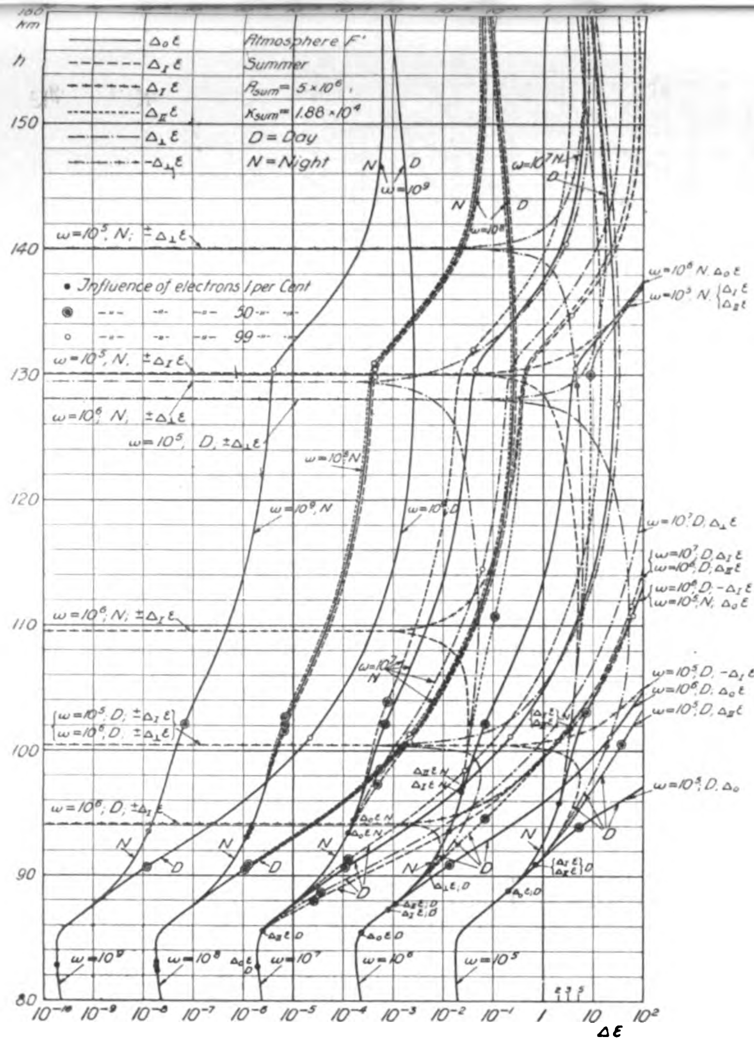

Fig. IX. 10. $\Delta_0\varepsilon$, $\Delta_I\varepsilon$, $\Delta_{II}\varepsilon$ and $\Delta_\perp\varepsilon$ as functions of h and ω from $h = 80$ km to $h = 160$ km. Atmosphere F' and summer time.

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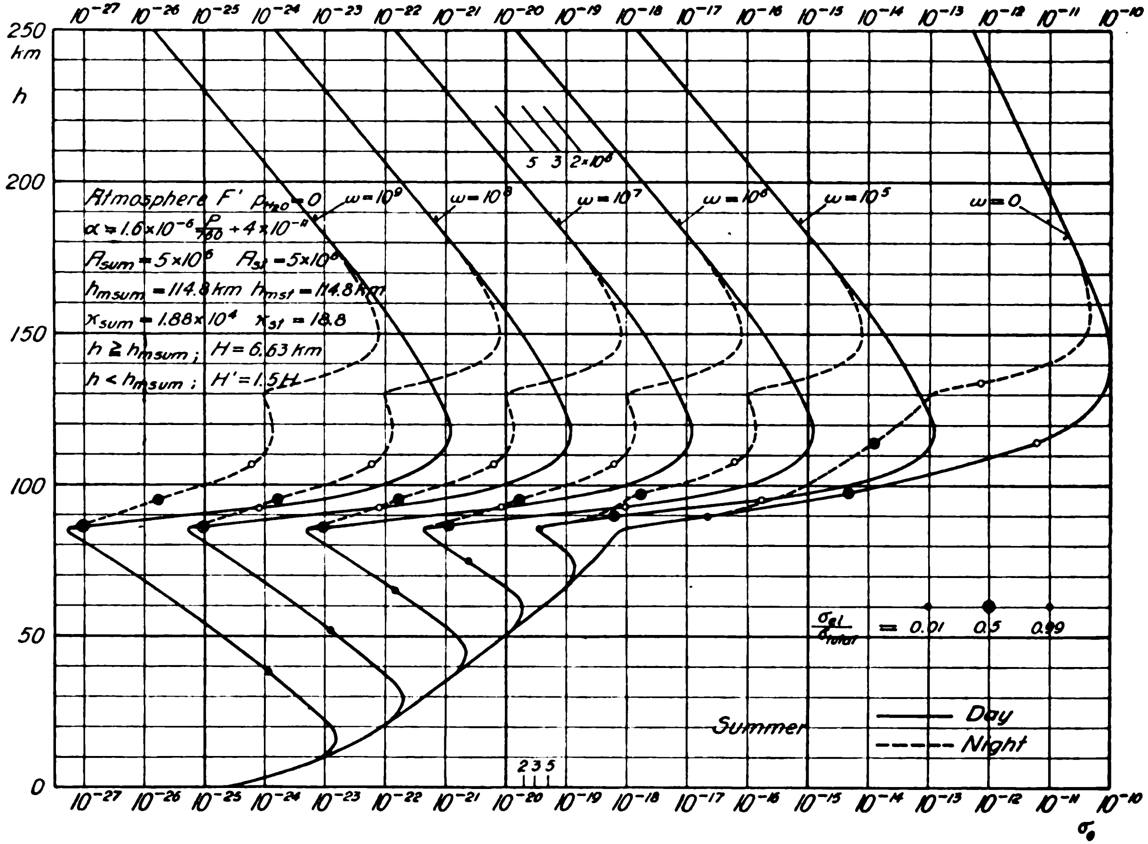

Fig. IX. 11. σ_0 as a function of h and ω from $h=0$ to $h=250$ km. Atmosphere F' and summer time.

Below 84 km the conductivity is mainly due to the ionization caused by the highly penetrating radiation. In this region σ_0 is taken as one fourth of the conductivity which would exist if all the ions remained mono-molecular (see p. 77).

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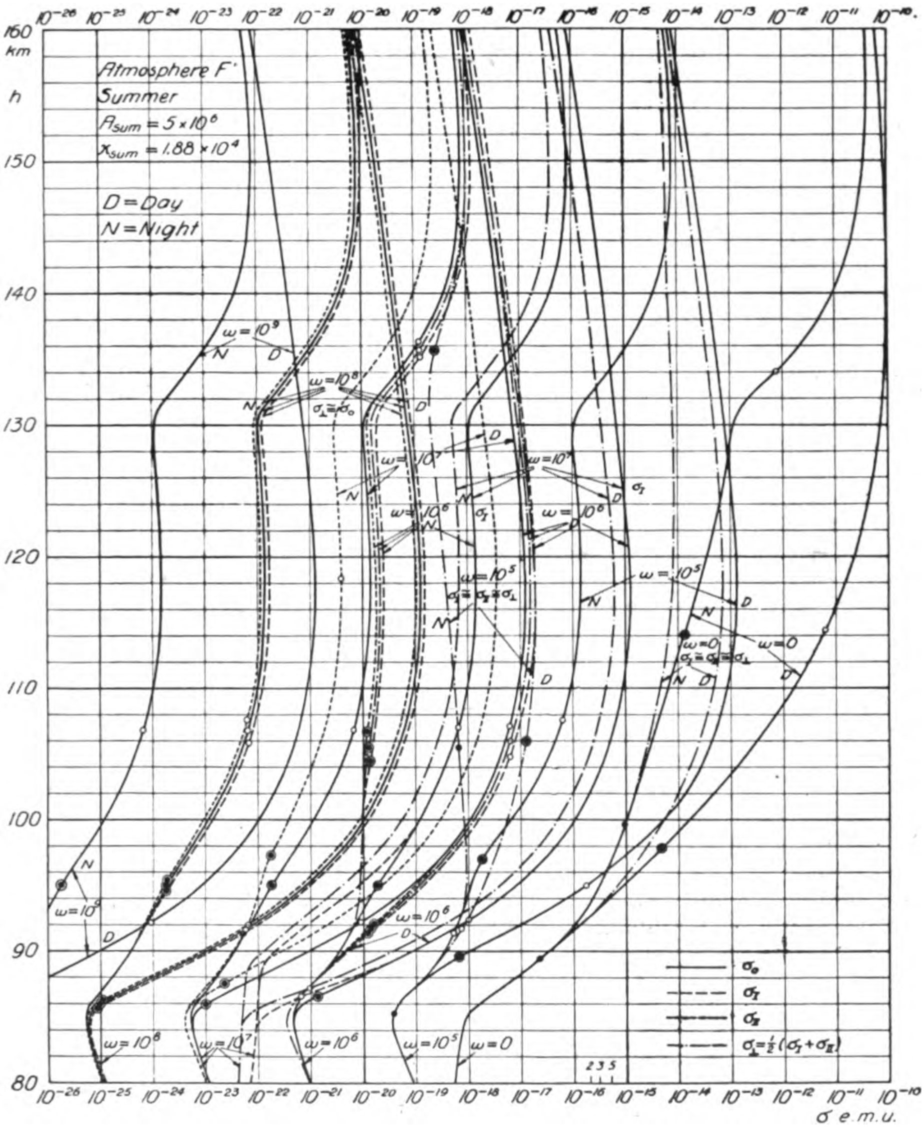

Fig. IX. 12. σ_0, σ_I, σ_{II} **and** $\sigma_\perp$ **as functions of** h **and** ω, **from** $h = 80$ **km to** $h = 160$ **km. Atmosphere** F' **and summer time.**

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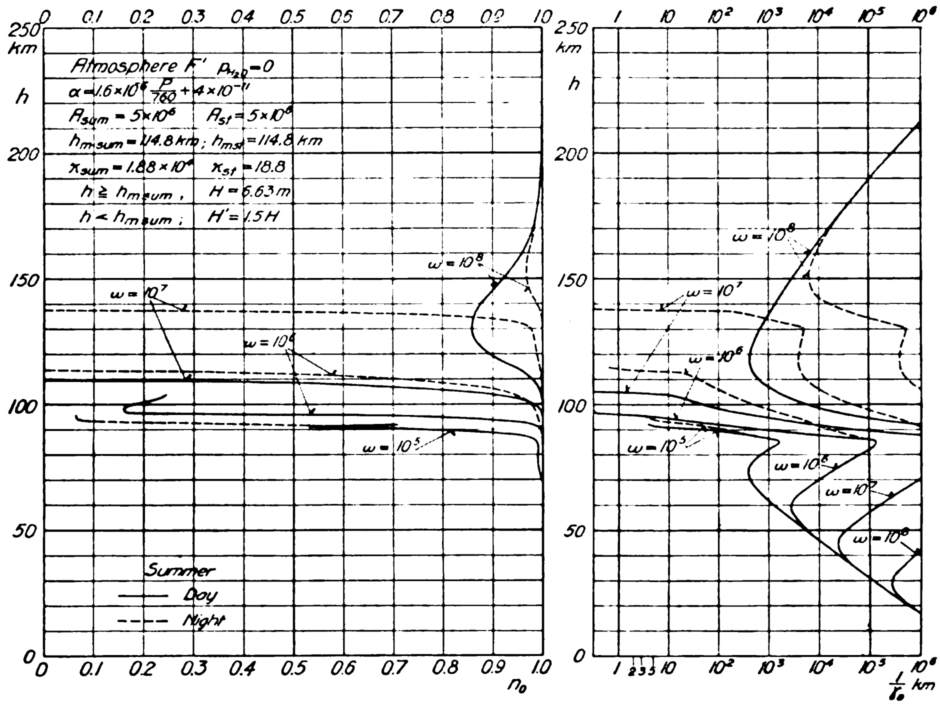

Fig. IX. 13. The refractive index n_0 and the attenuation constant γ_0 as functions of the frequency and of the altitude h, from $h=0$ to $h=250$ km. Atmosphere F' and summer time. No magnetic field.

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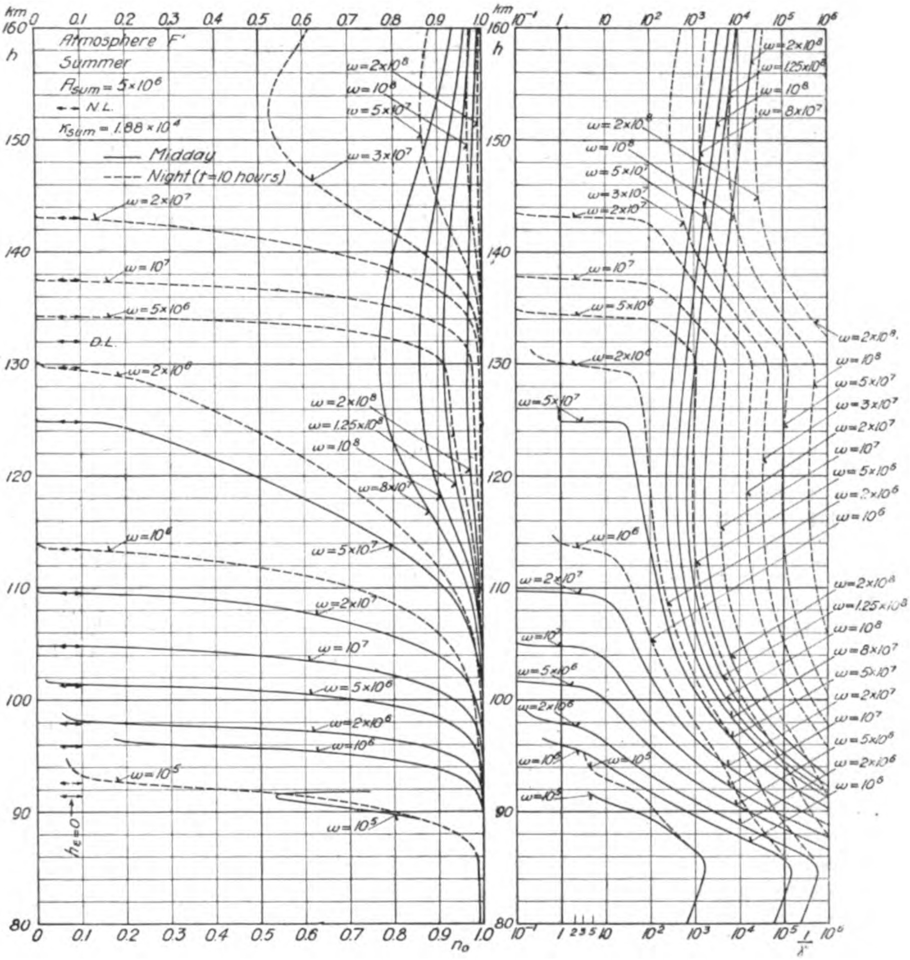

Fig. IX. 14. The refractive index n_0 and the attenuation constant γ_0 as functions of the frequency and of the altitude h, from $h = 80$ km to $h = 160$ km. Atmosphere F' and summer time. $h_{\varepsilon=0}$ denotes the altitudes at which $\varepsilon = 0$ (*i. e.* $\Delta\varepsilon = 1$).

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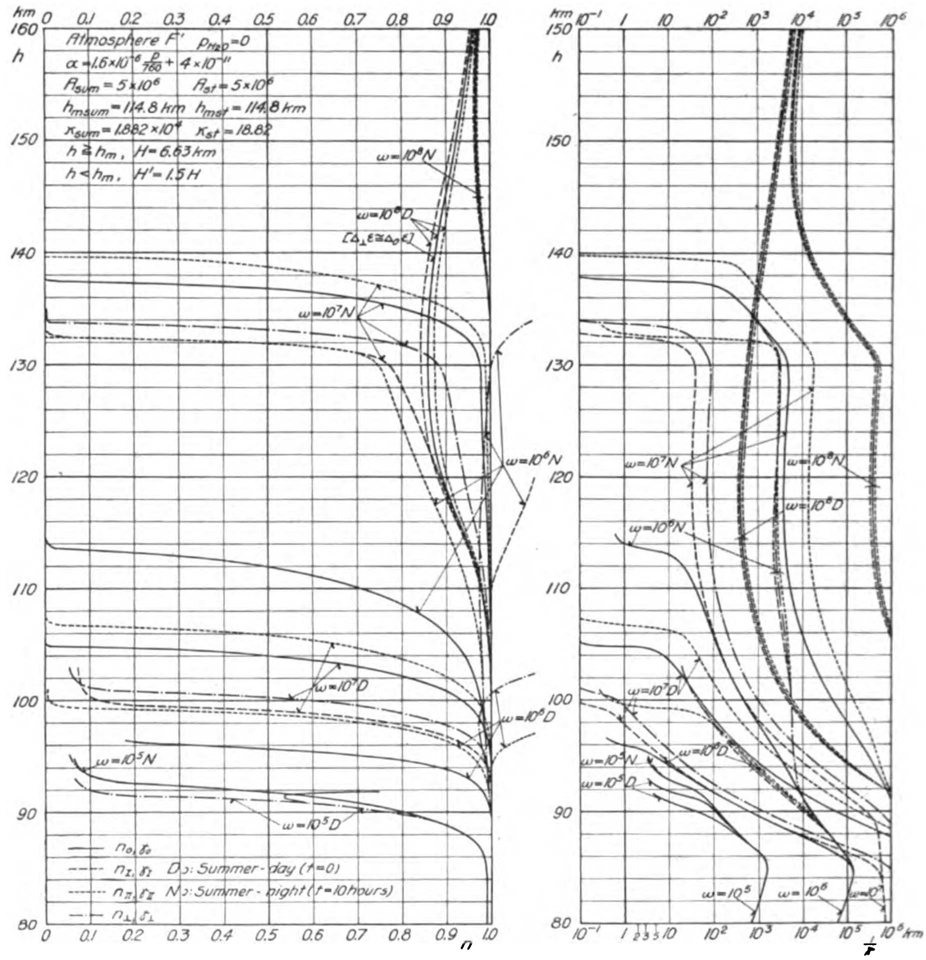

Fig. IX. 15. n_0, n_I, n_{II}, $n_\perp$, γ_0 γ_I, γ_{II} and $\gamma_\perp$ as functions of the frequency and of the altitude h from $h = 80$ km to $h = 160$ km. Atmosphere F' and summer time.

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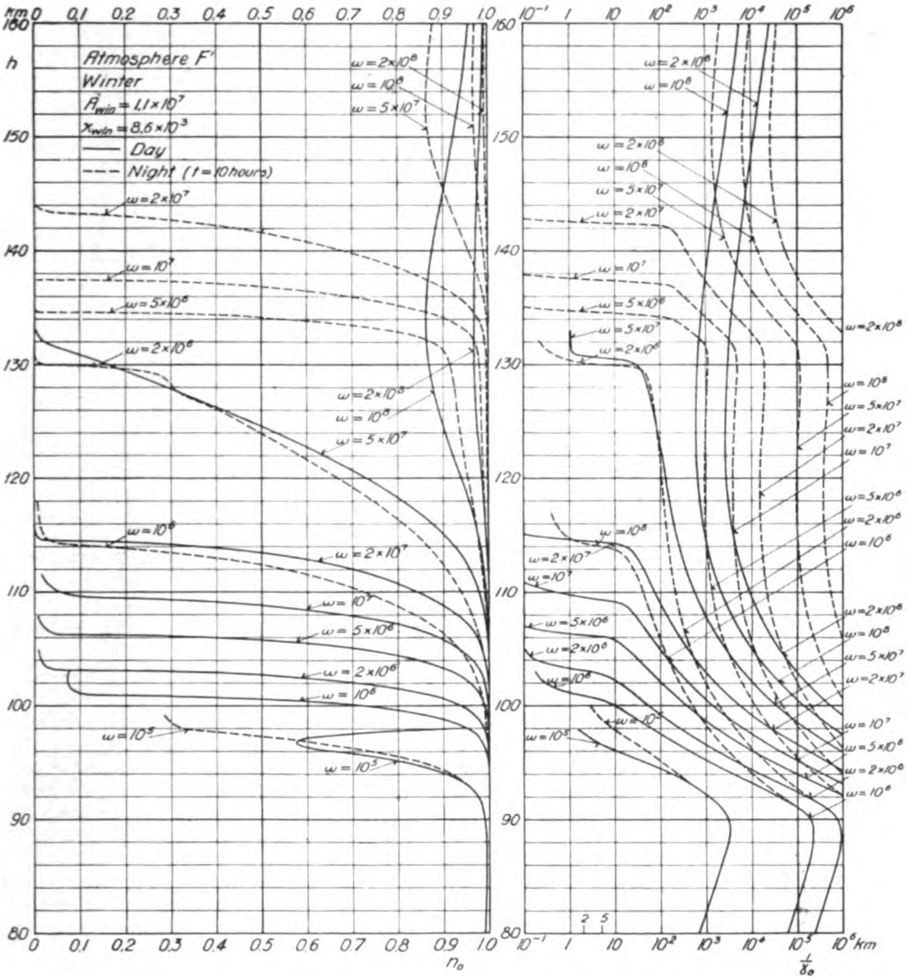

Fig. IX. 16. The refractive index n_0 and the attenuation constant γ_0 as functions of the frequency and of the altitude h, from $h = 80$ km to $h = 160$ km. Atmosphere F' and winter time.

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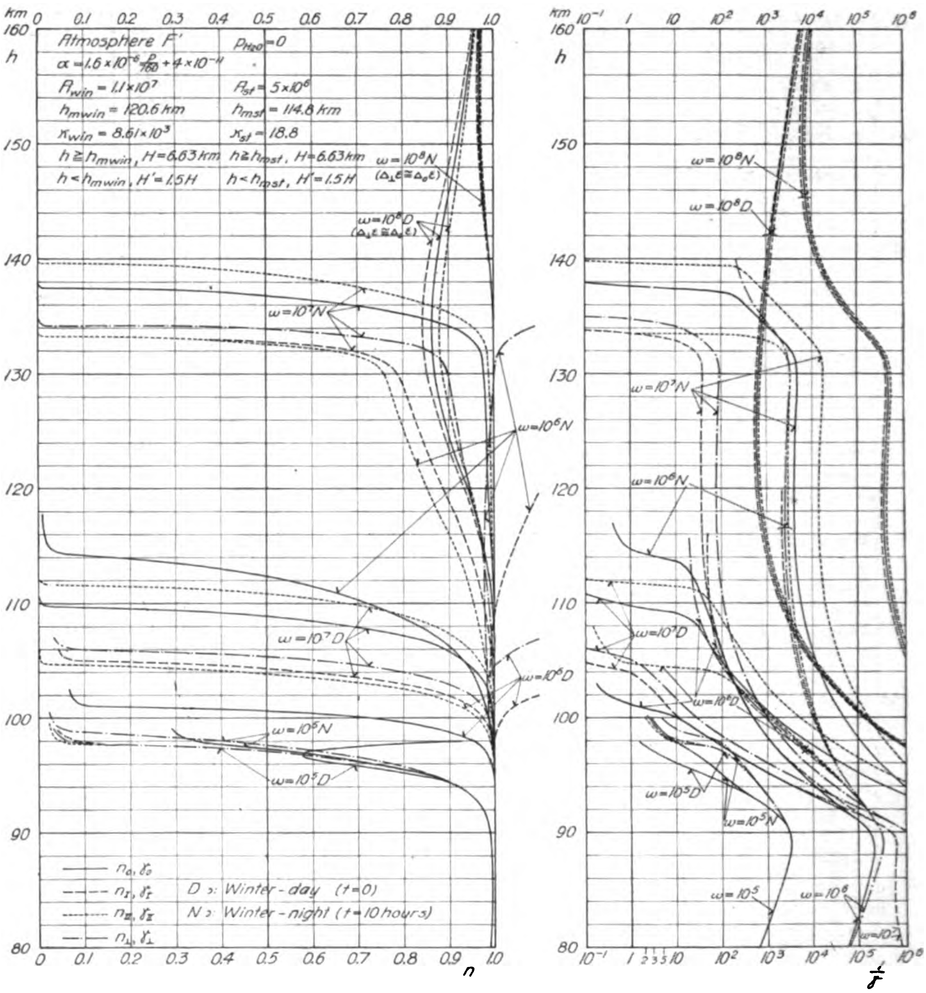

Fig. IX. 17. n_0, n_{I}, n_{II}, $n_\perp$, γ_0, γ_{I}, γ_{II} and $\gamma_\perp$ as functions of the frequency and of the altitude h from $h = 80$ km to $h = 160$ km. Atmosphere F' and winter time.

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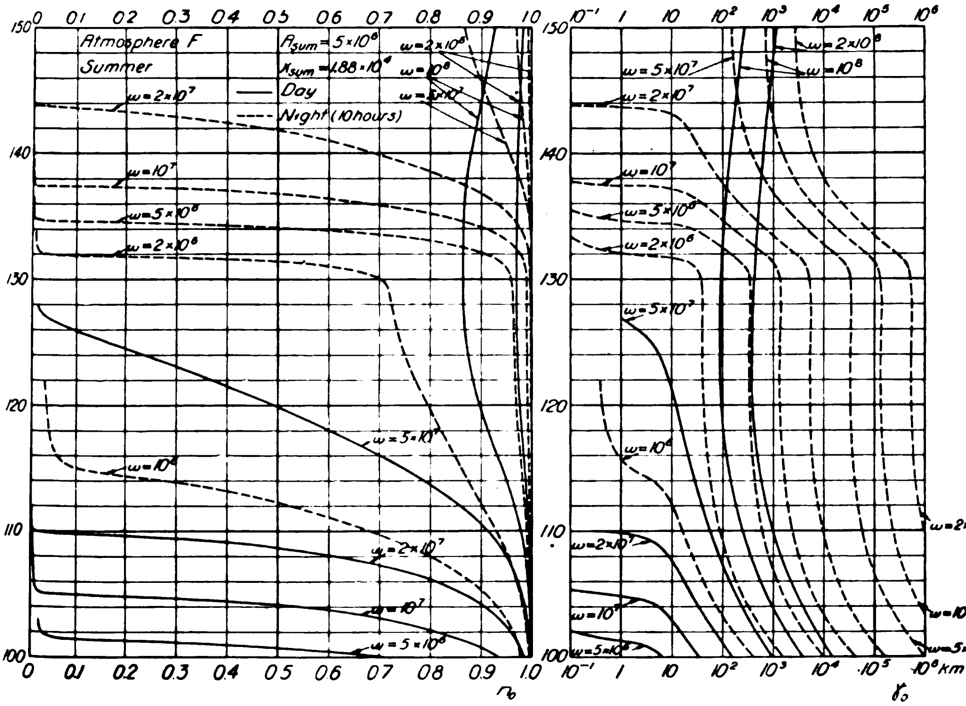

Fig. IX. 18. The refractive index n_0 and the attenuation constant γ_0 as functions of the frequency and of the altitude h from $h = 100$ km to $h = 150$ km. Atmosphere F and summer time.

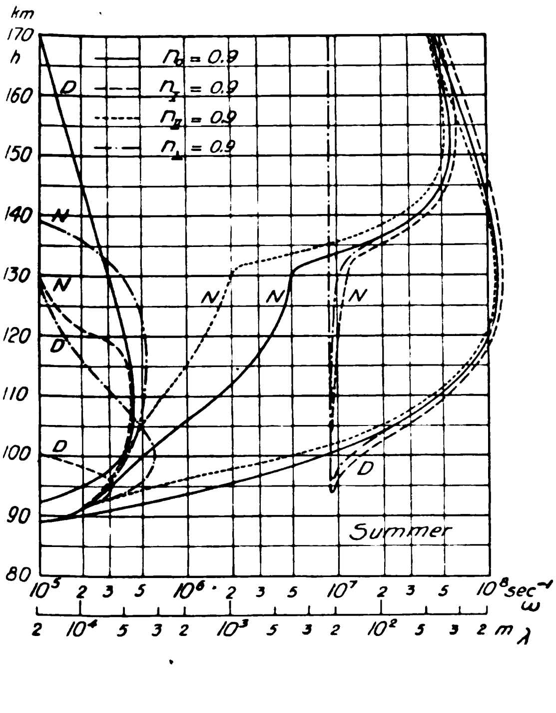

Fig. IX. 20. The altitude h at which $n = 0.9$ as a function of ω. Atmosphere F and summer time.

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